Where is the PdV term in the first law of black-hole thermodynamics?

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Outline

Temperature and Entropy

- Schwarzschild line element:

$$ds^2 = -\Delta(r)dt^2 + \frac{1}{\Delta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

$$\Delta(r) = \left(1 - \frac{2G_N M}{r}\right) \quad \Rightarrow \quad r_h = 2G_N M.$$

  - Area: $A = 16\pi G_N^2 M^2$.
- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$ ($\ell_{Pl}^2 = \hbar G_N$, $c = 1$). Bekenstein (1972)
- Temperature, $T = \frac{\kappa\hbar}{2\pi}$: κ =surface gravity. Hawking (1974)

Schwarzschild, $\kappa = \frac{1}{4G_N M}$

$$T = \frac{\hbar}{8\pi G_N M}.$$

Solar mass black hole: $T = 6 \times 10^{-8}$ K, $S \approx 10^{78}$.

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First and Second laws

- Internal energy $U(S)$, $T = \frac{\partial U}{\partial S}$: identify $M = U(S)$.

First Law of Black Hole Thermodynamics

$$dM = dU = T\,dS$$

- With $S = a\frac{A}{\hbar G_N}$
$r_h = 2G_N M$, $A = 16\pi G_N^2 M^2$, $dS = \frac{32\pi a G_N}{\hbar} M dM$, $T = \frac{\hbar}{8\pi G_N M}$,
$TdS = 4adM \quad \Rightarrow \quad a = \frac{1}{4}$.

Bekenstein-Hawking Entropy

$$S = \frac{1}{4}\frac{A}{\ell_{Pl}^2}.$$

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Hawking radiation

- Free energy, $F(T)$, is Legendre transform of $U(S)$

$$F = U - TS = M - \frac{\kappa A}{8\pi G_N}.$$

- Schwarzschild:

$$F = \frac{M}{2} = \frac{\hbar}{16\pi G T}.$$

  - Heat capacity:

$$C = \frac{\partial U}{\partial T} = -T\frac{\partial^2 F}{\partial T^2} = -\frac{8\pi G_N M^2}{\hbar} = -2S < 0.$$

    Negative Heat capacity!
  - Radiates with power $\mathcal{P} \sim \frac{AT^4}{\hbar^3} \sim \frac{\hbar}{G_n^2 M^2}$.
    Lifetime $\tau \sim \frac{M}{\mathcal{P}} \sim \frac{G_N^2 M^3}{\hbar}$, $M \sim 10^{12}\ kg \ \Rightarrow \tau \sim 10^{10}\ years$.
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Pressure and enthalpy

First Law of thermodynamics

$$dU = T\,dS \;\; - P\,dV$$

- Where is the $P\,dV$ term in the first law?
- Include cosmological constant Λ, contributes pressure P and energy density $\epsilon = -P = \frac{\Lambda}{8\pi G_N}$.
  Henneaux+Teitelboim (1984); (1989); Teitelboim (1985).
- Thermal energy

$$U = M + \epsilon V = M - PV \quad \Rightarrow \quad M = U + PV$$

$$U = U(S, V)$$

Enthalpy

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Kastor, Ray+Traschen [0904.2765].

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What is V?

- Include Λ,

$$\Delta(r) = 1 - \frac{2G_N M}{r} - \frac{\Lambda}{3} r^2 \qquad (\Lambda < 0),$$

$$\Delta(r_h) = 0 \;\Rightarrow\quad M(r_h, \Lambda) = H(S, P), \qquad S = \frac{\pi r_h^2}{\ell_{Pl}^2},\; P = \frac{\Lambda}{8\pi G_N}.$$

- Thermodynamic definition of V (BPD [1008.5023]):

$$V = \left(\frac{\partial H}{\partial P}\right)_S \quad\Rightarrow\quad V = \frac{4}{3}\frac{(\ell_{Pl} S)^{\frac{3}{2}}}{\sqrt{\pi}} = \frac{4\pi r_h^3}{3}.$$

- Problem $S = \frac{\pi r_h^2}{\ell_{Pl}^2}$ and $V = \frac{4\pi r_h^3}{3}$ are not independent!
- Resolved by introducing rotation, J.

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AdS-Kerr Thermodynamics

- Rotating black-hole in asymptotically AdS (Kerr-AdS).
- Angular momentum $J = aM$ $(G_N = \hbar = 1, \Lambda = -\frac{3}{L^2})$
- Area of event horizon: $A = \frac{4\pi(r_h^2 + a^2)}{1 - \frac{a^2}{L^2}} = 4S$.
- Mass and enthalpy

$$M = \frac{1}{2}\sqrt{\frac{\left(1 + \frac{8PS}{3}\right)\left\{S^2\left(1 + \frac{8PS}{3}\right) + 4\pi^2 J^2\right\}}{\pi S}} := H(S, P, J)$$

Sorkin (1982); Caldarelli, Gognola+Klemm [hep-th/9908022].

- Thermodynamic volume:

$$V = \left.\frac{\partial H}{\partial P}\right|_{S,J} = \frac{2}{3\pi H}\left\{S^2\left(1 + \frac{8PS}{3}\right) + 2\pi^2 J^2\right\}$$

Cvetic, Gibbons+Kubizňák [1012.2888]; BPD [1106.6260].

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$$V = \left.\frac{\partial H}{\partial P}\right|_{S,J} = \frac{2}{3\pi H}\left\{S^2\left(1 + \frac{8PS}{3}\right) + 2\pi^2 J^2\right\}$$

Cvetic, Gibbons+Kubizňák [1012.2888]; BPD [1106.6260].

-

$$\left(\frac{3V}{4\pi}\right)^{1/3} \geq \left(\frac{S}{\pi}\right)^{1/2}$$

Equality only for $J = 0$.

AdS-Kerr Thermodynamics

- Rotating black-hole in asymptotically AdS (Kerr-AdS).
- Angular momentum $J = aM$ $(G_N = \hbar = 1, \Lambda = -\frac{3}{L^2})$
- Area of event horizon: $A = \frac{4\pi(r_h^2+a^2)}{1-\frac{a^2}{L^2}} = 4S$.
- Mass and enthalpy

$$M = \frac{1}{2}\sqrt{\frac{\left(1+\frac{8PS}{3}\right)\left\{S^2\left(1+\frac{8PS}{3}\right)+4\pi^2 J^2\right\}}{\pi S}} := H(S,P,J)$$

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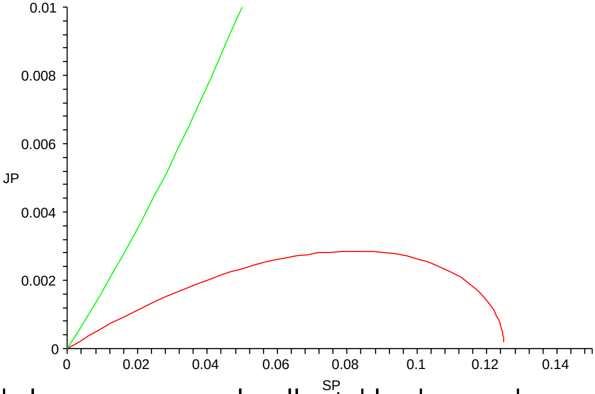
Critical behaviour



- $T > 0$ below green curve, locally stable above red curve.
- $C_P \to \infty$ on red curve: two values of SP for given JP ('small' and 'large' black holes), merge to one at critical point $(JP)_{crit}$, $(SP)_{crit}$. (Caldarelli, Gognola+Klemm [hep-th/9908022])

$(SP)_{crit} \approx 0.08204$, $(JP)_{crit} \approx 0.002857$, $(VP^{\frac{3}{2}})_{crit} \approx 0.01768$.

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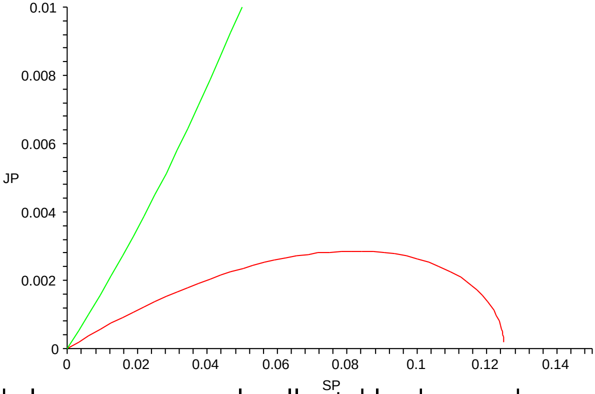

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Equation of state and critical exponents

- Define
$$t := \frac{T - T_c}{T_c}, \qquad v := \frac{V - V_c}{V_c}, \qquad p := \frac{P - P_c}{P_c}.$$
Expand the equation of state about the critical point:
$$p = 2.42t - 0.81tv - 0.21v^3 + o(t^2, tv^2, v^4).$$
cf. van der Waals gas: $p = 4t - 6tv - \frac{3}{2}v^3 + o(t^2, tv^2, v^4)$.
- $C_V = T / \left.\frac{\partial T}{\partial S}\right|_{V,J} \propto t^{-\alpha}$;
- At fixed $p < 0$, $v_> - v_< \propto |t|^\beta$;
- Isothermal compressibility, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,J} \propto t^{-\gamma}$;
- At $t = 0$, $|p| \propto |v|^\delta$;

Mean Field Exponents

$$\alpha = 0, \qquad \beta = \frac{1}{2}, \qquad \gamma = 1, \qquad \delta = 3.$$

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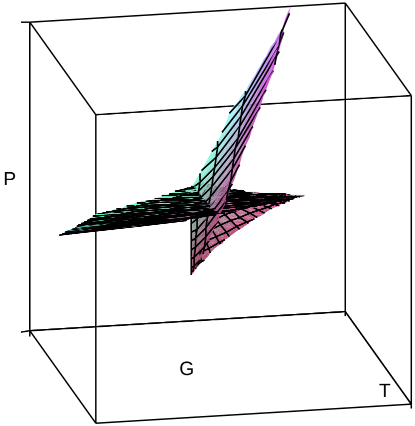
Gibbs Free Energy, $G(T, P, J) = H(S, P, J) - TS$: $(J = 1)$



Kerr-Reissner-Nordström-AdS

Similar story for $J = 0, Q \neq 0$ (Reissner-Nordström – AdS).

- Gibbs free energy is a 'swallowtail', same as van der Waals Champlin, Emparan, Johnson+Myers [hep-th/9902170;9904197].
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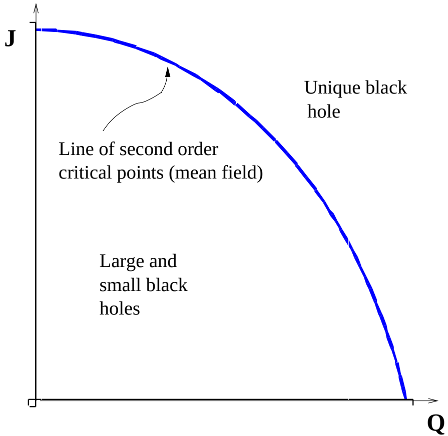


Conclusions

- $\Lambda \neq 0 \quad \Rightarrow \quad P\,dV$ term in black hole 1st law.

First Law

$$dU = TdS + \Omega dJ + \Phi dQ - PdV$$

- Black hole mass is identified with **enthalpy**, $H(S, P)$:

$$dM = dH = T\,dS + \Omega dJ + \Phi dQ + V\,dP.$$

- "Thermodynamic" volume: $V = \left(\frac{\partial H}{\partial P}\right)_S$.

- When $J \to 0$, V and S are not independent,

$$\left(\frac{3V}{4\pi}\right)^2 \geq \left(\frac{S}{\pi}\right)^3 ,$$

reverse iso-perimetric inequality.
- Gibbs free energy: $G(T, P) = -T \ln Z$, $(Z = e^{-I_E})$.
- Line of second order critical points in $J - Q$ plane with mean field exponents and van der Waals type equation of state.

Outlook

- Heat capacity: $C_P = \frac{T}{\frac{\partial T}{\partial S}\big|_P}$; $C_V = \frac{T}{\frac{\partial T}{\partial S}\big|_V}$.
  Condensed matter applications?
- Higher dimensional rotating black holes; other backgrounds?
- de Sitter, $\Lambda > 0$, $P < 0$?
- Beyond mean field?

Smarr relation

- Ordinary thermodynamics: $U(S,V,n)$ (n = number of moles) is a function of extensive variables. U is also extensive $\Rightarrow$

$$\begin{aligned}
\lambda^d U(S,V,n) &= U(\lambda^d S, \lambda^d V, \lambda^d n) \\
\Rightarrow \qquad U &= S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n} && \text{Euler equation} \\
\Rightarrow \qquad U &= ST - VP + n\mu && (\mu = \text{chemical potential}) \\
\Rightarrow \qquad G &= U + VP - ST = n\mu\,. && \text{Gibbs-Duhem relation}
\end{aligned}$$

- Black hole: $S \to \lambda^2 S$, $P \to \lambda^{-2} P$, $J \to \lambda^2 J$, $M \to \lambda M \quad \Rightarrow$

$$\begin{aligned}
\lambda H(S,P,J) &= H(\lambda^2 S, \lambda^{-2} P, \lambda^2 J) && M = H(S,P,J) \\
H &= 2S\frac{\partial H}{\partial S} - 2P\frac{\partial H}{\partial P} + 2J\frac{\partial H}{\partial J} \\
\Rightarrow \qquad H &= 2ST - 2VP + 2J\Omega && \text{Smarr relation}
\end{aligned}$$

Smarr (1973); Kastor, Ray+Traschen [0904.2765].

Penrose Processes

Mechanical energy:

$$dW = -dU = -TdS - \Omega dJ - \Phi dQ + PdV.$$

- Most efficient is $S = const$ and decrease $|J|$ and $|Q|$.
- Maximum work starting from an extremal black-hole.
- Start from $J = J_{max}$, $Q = 0$ and keep P constant $\Rightarrow$ maximum efficiency: $\eta_{max} = 51.8\%$, for $PS = 1.837$
  BPD [1106.6260]
  (compare $\eta_{max} = 1 - \frac{1}{\sqrt{2}} = 29\%$, *e.g.* Wald (1984)).
- Charged-Kerr-AdS: $\eta_{max} = 75\%$ (compare $\eta_{max} = 50\%$).

- Adiabatic compressibility: $\kappa = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_{S,J} \geq 0.$
- $\kappa_{J=0} = 0.$
- Maximum for J_{max} $(T = 0)$: $\kappa_{max} = \frac{2S(1+8PS)}{(3+8PS)^2(1+4PS)}.$
- e.g. $P = 0$,

  $\kappa_{max} = \frac{2S}{9} = \frac{4\pi M^2}{9} = 2.6 \times 10^{-38} \left(\frac{M}{M_\odot}\right)^2 \; m\, s^2\, kg^{-1}.$

  cf. neutron star, $M \approx M_\odot$, $R \approx 10\, km$, degenerate Fermi gas $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}.$

  Very stiff equation of state!
- $\rho = \frac{M}{V}$, "speed of sound" $v_s^{-2} = \left.\frac{\partial \rho}{\partial P}\right|_{S,J}$,

"Speed of Sound"

$$v_s^{-2} = 1 + \frac{(2\pi J)^4}{\left(2S^2 + (2\pi J)^2\right)^2}$$

$\Rightarrow \quad \frac{1}{2} \leq v_s^2 \leq 1$, with $v_s = 1$ for $J = 0$.

Internal Energy

$U = H + PV$

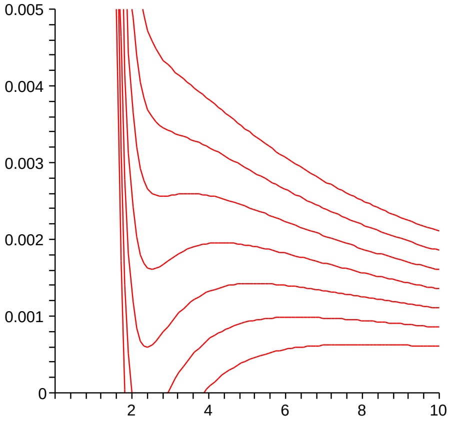
$$U(S,V,J) = \left(\frac{\pi}{S}\right)^3 \left\{ \left(\frac{3V}{4\pi}\right)\left(\frac{S^2}{2\pi^2} + J^2\right) - J^2 \sqrt{\left(\frac{3V}{4\pi}\right)^2 - \left(\frac{S}{\pi}\right)^3} \right\},$$

BPD [1106.6260].

- As $J \to 0$, $\left(\frac{3V}{4\pi}\right)^2 \to \left(\frac{S}{\pi}\right)^3$.
  $T = \frac{\partial U}{\partial S}\big|_{V,J}$ finite $\quad \Rightarrow \quad \lim_{J\to 0} \frac{J^2}{\sqrt{\left(\frac{3V}{4\pi}\right)^2 - \left(\frac{S}{\pi}\right)^3}}$ is finite.
  But $\lim_{J\to 0} \frac{\partial^2 U}{\partial S^2} \to \infty \quad \Rightarrow \quad C_V = \frac{\partial U}{\partial T}\big|_V = T\,\frac{\partial S}{\partial T}\big|_{V,J} \to 0$ as $J \to 0$.
- $P = -\frac{\partial U}{\partial V}\big|_{S,J}$. Virial expansion at fixed T (large V),

$$P = \frac{T}{2}\left(\frac{4\pi}{3V}\right)^{1/3} - \frac{1}{8\pi}\left(\frac{4\pi}{3V}\right)^{2/3} + \frac{4\pi}{3}\frac{J^2}{V^2} + o\left(\frac{J^4}{V^{11/3}}\right).$$

$P - V$ diagram

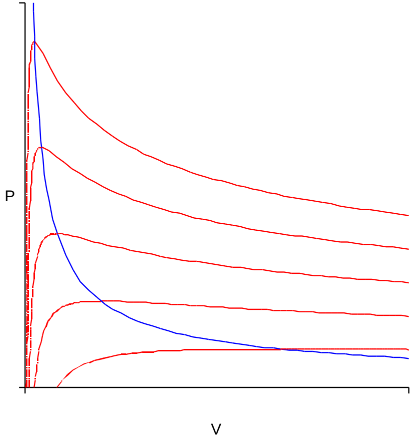


P as a function of $\left(\frac{3V}{4\pi}\right)^{1/3}$, curves of constant T for $J = 1$.
Critical point at $T_c = 0.0418\, J^{1/2}$, $P_c = 0.00286/J$ and $V_C = 115.8\, J^{3/2}$. (Caldarelli, Gognola+Klemm [hep-th/9908022]).

Equation of state

- Equation of state

$$T(V,P) = \frac{\hbar}{4\pi}\left\{\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}} + 8\pi GP\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}\right\}$$



BTZ black hole

$$d^2s = -\Delta(r)dt^2 + \Delta^{-1}(r)dr^2 + r^2 d\phi^2,$$

$$\Delta(r) = -8G_N M + \frac{r^2}{L^2}. \qquad (\Lambda = -\tfrac{1}{L^2})$$

- $r_h = \sqrt{8G_N M}L, \quad S = \frac{\pi r_h}{2\ell_{Pl}}$ $\qquad (\ell_{Pl} = \hbar G_N).$
- Enthalpy $H = M$:
$$H(S,P) = \frac{4\ell_{Pl}}{\pi}S^2 P.$$
- Equation of state:
$$PV^{\frac{1}{2}} = \frac{\sqrt{\pi}}{4\ell_{Pl}}T.$$
- Thermodynamic volume:
$$V = \pi r_h^2.$$
- Gibbs free energy:
$$G = H - TS = -M.$$

- Heat capacity:

$$C_P = \frac{T}{\frac{\partial T}{\partial S}\Big|_P} = S > 0.$$

- Phase transition. Compare pure AdS_3 with

$$\Delta(r) = 1 + \frac{r^2}{L^2}$$

(Equivalent to setting $M = -\frac{1}{8G_N}$ in BTZ).

- Enthalpy, $H = -\frac{1}{8G_N}$ = constant, $T = 0 \Rightarrow$ Gibbs free energy:

$$G = H = -\frac{1}{8G_N}.$$

- If $M < \frac{1}{8G_N}$ in BTZ, pure AdS_3 has lower free energy.
- Equivalent to temperature: $T = \hbar\sqrt{\frac{2G_N P}{\pi}}$.

Quantum corrections to BTZ equation of state

(Brown+Henneaux (1984); Maloney+Witten [0712.0155])

- Rotating BTZ black hole: (angular momentum J).

$$ds^2 = -\Delta(r)dt^2 + \frac{1}{\Delta(r)}dr^2 + r^2\left(d\phi - \frac{4G_N J}{r^2}dt\right)^2$$

$$\Delta(r) = \left(-8G_N M + \frac{r^2}{L^2} + \frac{16G_N^2 J^2}{r^2}\right)$$

- Inner and outer event horizons:

$$r_\pm^2 = 4G_N ML^2\left\{1 \pm \left[1 - \left(\frac{J}{ML}\right)^2\right]^{\frac{1}{2}}\right\}.$$

- Hawking temperature:

$$T = \frac{\Delta'(r_h)}{4\pi} = \frac{(r_h^2 - r_-^2)\hbar}{2\pi L^2 r_h}.$$

Partition function

- Wick rotate to Euclidean time: $t \to -it_E$, $J \to iJ_E$.
  Also $r_h \to r_{E,+}$ and $r_- \to ir_{E,-}$ where

$$r_{E,\pm}^2 = 4G_N ML^2 \left\{ \left[1 + \left(\frac{J_E}{ML} \right)^2 \right]^{\frac{1}{2}} \pm 1 \right\}.$$

- Define

$$\tau = \frac{r_{E,-} + ir_{E,+}}{L} \qquad (Im(\tau) > 0).$$

  Inverse Hawking temperature:

$$\frac{1}{2\pi T} = \frac{r_{E,+}}{(r_{E,+}^2 + r_{E,-}^2)} \frac{L^2}{\hbar} = \frac{L}{\hbar} Im \left(-\frac{1}{\tau} \right).$$

- Define $q = e^{2\pi i \tau}$ in terms of which

$$Z_{BTZ} = (q\bar{q})^{-\frac{L}{16\hbar G_N}} \prod_{n=2}^{\infty} |1 - q^n|^{-2}.$$

  to all orders in perturbation theory.

Entropy corrections

Set $J = 0$: $T = \frac{r_h}{2\pi}\frac{\hbar}{L^2}$, $\quad \tau = i\frac{r_h}{L} \quad$ and $\quad q = e^{-4\pi^2 \frac{LT}{\hbar}}$.

- The $J = 0$ partition function:

$$Z_{BTZ} = e^{\frac{\pi^2 TL^2}{2\hbar^2 G_N}} \prod_{n=2}^{\infty} \left(1 - e^{-4\pi^2 n \frac{TL}{\hbar}}\right)^{-2}.$$

- Defining $x = \frac{TL}{\hbar} = \frac{r_h}{2\pi L}$, the Gibbs free energy is

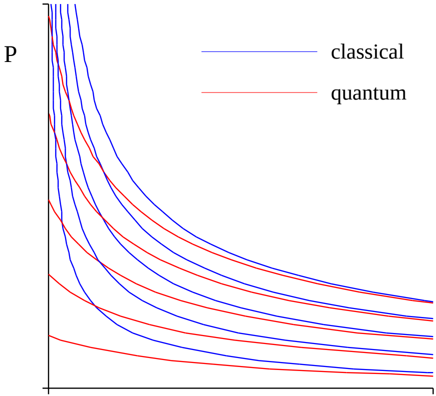
$$G(T,P) = -T \ln Z_{BTZ} = -\frac{\pi^2 x^2}{2G_N} + 2T \sum_{n=2}^{\infty} \ln\left(1 - e^{-4\pi^2 nx}\right)$$

- Entropy, $S < \frac{1}{4} \times (\text{area})$.

- Equation of state (quantum volume):

$$V(T,P) = \left.\frac{\partial G}{\partial P}\right|_T = \pi r_h^2 \left[1 - 8\pi G_N \frac{\hbar}{L} \sum_{n=2}^{\infty} \frac{n}{e^{4\pi^2 n x} - 1}\right]$$

(BPD [arXiv:1008.5023]).



Higher dimensional black holes

- Line element: $d^2s = -\Delta(r)dt^2 + \Delta^{-1}(r)dr^2 + r^d d^2\Omega_{(d)}$.
  $\Omega_{(d)} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$: volume of d-dimensional unit sphere.
- Generalisation: event horizon any d-dimensional Einstein space

$$R_{ij} = \frac{R}{d} g_{ij}$$

  with constant Ricci curvature R and unit radius volume $\Omega_{(d)}$
  (e.g. flat torus, or $CP^{\frac{d}{2}}$ for even d).

$$\Delta(r) = \frac{R}{d(d-1)} - \frac{16\pi G_N}{\Omega_{(d)}\, d} \frac{M}{r^{d-1}} - \frac{2\Lambda}{d(d+1)} r^2$$

- Planck length $\ell_{Pl}^d = \hbar G_N$:

$$S = \frac{\Omega_{(d)}}{4} \frac{r_h^d}{\ell_{Pl}^d}, \qquad P = -\frac{\Lambda}{8\pi G_N}.$$

- Identify $H(S,P) = M \;\Rightarrow$

$$H(S,P) = \frac{\hbar S}{4\pi}\left\{ \frac{R}{d-1}\left(\frac{4\ell_{Pl}^d S}{\Omega_{(d)}}\right)^{-\frac{1}{d}} + \frac{16\pi G_N P}{d+1}\left(\frac{4\ell_{Pl}^d S}{\Omega_{(d)}}\right)^{\frac{1}{d}} \right\},$$

gives thermodynamic volume

$$V = \left(\frac{\partial H}{\partial P}\right)_S = \frac{\Omega_{(d)} r_h^{d+1}}{d+1}$$

(BPD [arXiv:1008.5023]).

Equation of state

$$T = \frac{\hbar}{4\pi d}\left\{ R\left(\frac{(d+1)V}{\Omega_{(d)}}\right)^{-\frac{1}{d+1}} + 16\pi G_N P\left(\frac{(d+1)V}{\Omega_{(d)}}\right)^{\frac{1}{d+1}} \right\}$$

- Heat capacity

$$C_P = Sd \left\{ \frac{16\pi G_N P \left(\frac{4\ell_{Pl}^d S}{\Omega_{(d)}}\right)^{\frac{2}{d}} + R}{16\pi G_N P \left(\frac{4\ell_{Pl}^d S}{\Omega_{(d)}}\right)^{\frac{2}{d}} - R} \right\},$$

- For $R > 0$, phase transition at

$$T = \frac{2\hbar}{d}\sqrt{\frac{R_{(d)} G_N P}{\pi}}.$$

- For flat event horizon, $R = 0$,

$$C_P = Sd.$$