Localization of maximal entropy random walk

Z. Burda, J. Duda, J.-M. Luck, B. Waclaw

PRL 102, 160602 (2009)
APP B41, 949 (2010)

7th Vienna CE Seminar, Nov 2010

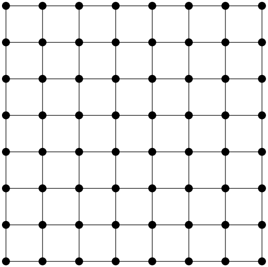
Outline

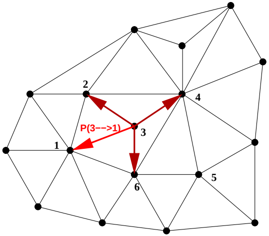
- Introduction/motivation;
- Generic Random Walk;
- Entropy;
- Maximal Entropy Random Walk;
- Localization
- Summary and open problems

Introduction

- Random walk on graphs;
- Local and global definitions;
- Global definition $\longrightarrow$ Maximal Entropy Random Walk;

RW on a finite connected graph





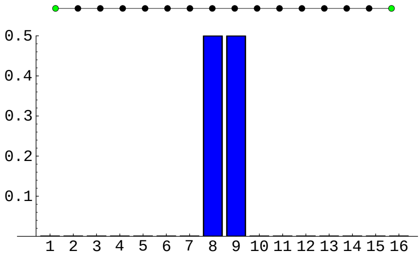
- Adjacency matrix: $A_{ij} = 1(0)$ if ij are (not) neighbors;
- Node degree: $k_i = \sum_j A_{ij}$;
- Hopping probability: $P(i \to j) = P_{ij}$
- Stochastic matrix $0 \leq P_{ij} \leq A_{ij}$; $\quad \sum_j P_{ij} = 1$;

Occupation probability

- Master equation: $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state: $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:

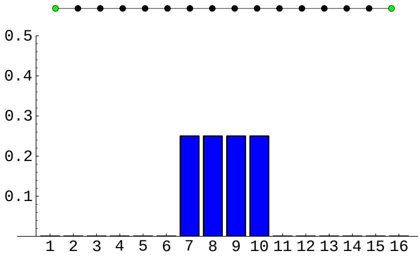


Occupation probability

- Master equation: $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state: $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:

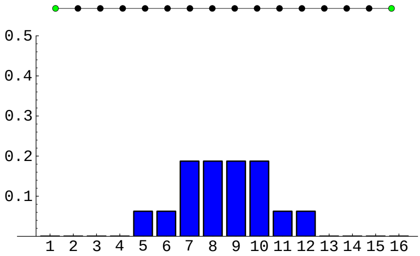


Occupation probability

- Master equation: $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state: $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:

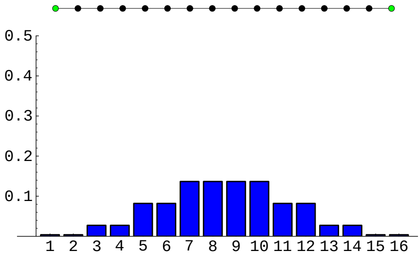


Occupation probability

- Master equation: $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state: $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:

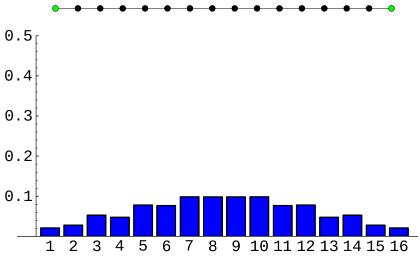


Occupation probability

- Master equation: $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state: $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:

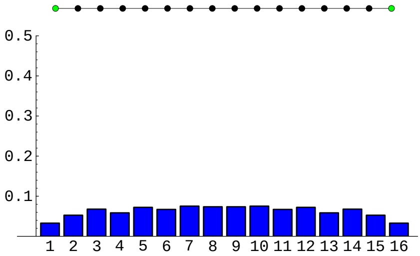


Occupation probability

- Master equation: $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state: $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:

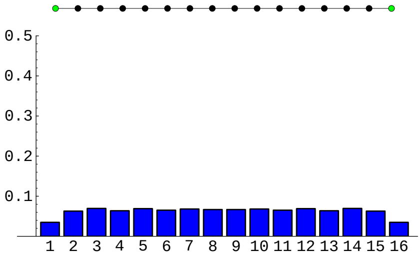


Occupation probability

- Master equation: $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state: $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:

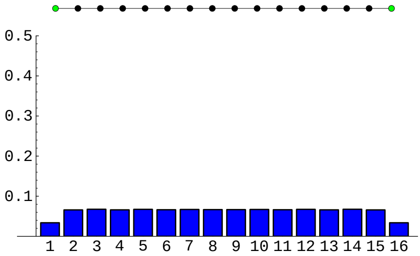


Occupation probability

- Master equation: $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state: $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:

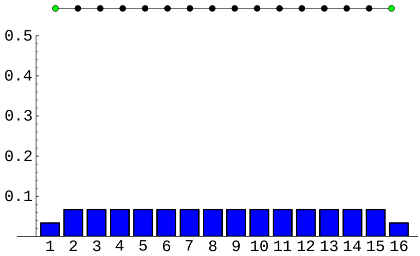


Occupation probability

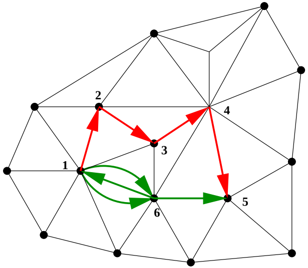
- Master equation: $\pi_i(t+1) = \sum_j \pi_j(t) P_{ji}$
- Stationary state: $\pi_i^* = \sum_j \pi_j^* P_{ji}$
- Generic Random Walk:

$$P_{ij} = \frac{A_{ij}}{k_i} \longrightarrow \pi_i^* = \frac{k_i}{\sum_j k_j}$$

- Example:



Random trajectories $\gamma_{ab}^{(t)}$



- Probability: $P(\gamma_{i_0 i_t}^{(t)}) = P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{t-1} i_t}$;
- GRW: $P(\gamma_{i_0 i_t}^{(t)}) = (k_{i_0} k_{i_1} \dots k_{i_{t-1}})^{-1}$;
- Example:
  $P(\gamma_{red}) = (7 \cdot 5 \cdot 4 \cdot 8)^{-1} = 1/1120$;
  $P(\gamma_{green}) = (7 \cdot 6 \cdot 7 \cdot 6)^{-1} = 1/1764$;
- MERW: $P(\gamma_{ab}^{(t)}) = F(t, a, b)$?

Entropy of random trajectories

- $S_t = -\sum_{\{\gamma_{ab}^{(t)}\}} P(\gamma_{ab}^{(t)}) \ln P(\gamma_{ab}^{(t)})$
- Entropy production rate (Shannon, McMillan):

$$s \equiv \lim_{t\to\infty} \frac{S_t}{t} = -\sum_i \pi_i^* \sum_j P_{ij} \ln P_{ij}$$

- $s_{\mathrm{GRW}} = \frac{\sum_i k_i \ln k_i}{\sum_i k_i} = \langle \ln k_i \rangle_*$
- Maximal entropy:

$$s_{\mathrm{max}} = \frac{\ln N_t}{t} = \frac{\ln (A^t)_{ab}}{t} \sim \ln \lambda_{max}$$

- Inequality:

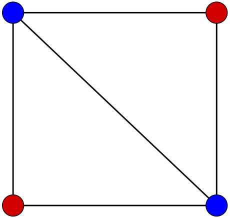
$$s_{\mathrm{GRW}} \leq s_{\mathrm{max}}$$

Frobenius-Perron eigenvalue

- FP: $k_{min} \leq \lambda_{max} \leq k_{max}$
- New inequality:

$$\exp\langle \ln k_i \rangle_* \leq \lambda_{max}$$

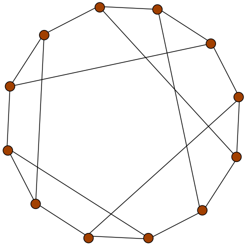
- Example:

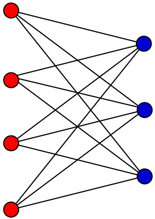


$\exp\langle \ln k_i \rangle_* = 108^{1/5} \approx 2.55085$;
$\lambda_{max} = (1 + \sqrt{17})/2 \approx 2.56155$;

Equality $s_{GRW} = s_{max}$

- $\exp\langle \ln k_i \rangle_* = \lambda_{max}$;
- k-regular graphs; $\lambda_{max} = k$;
- bi-regular bipartite graphs: $\lambda_{max} = \sqrt{k_1 k_2}$;





MERW

- Let $\sum_i \psi_i^2 = 1$ and

$$\sum_j A_{ij}\psi_j = \lambda_{max}\psi_i$$

- Transition probability:

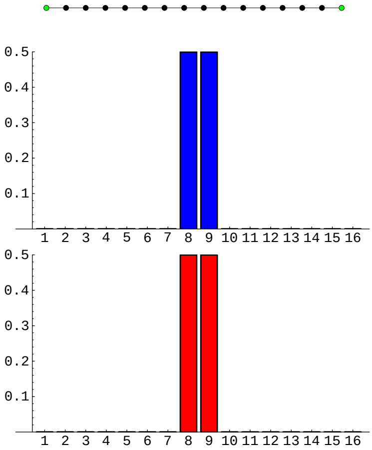
$$P_{ij} = \frac{A_{ij}}{\lambda_{max}} \frac{\psi_j}{\psi_i}$$

- Trajectories:

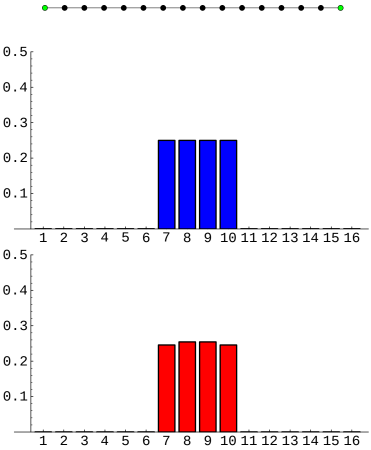
$$P(\gamma_{ab}^{(t)}) = \frac{1}{\lambda_{max}^t} \frac{\psi_b}{\psi_a}$$

- Stationary distribution: $\boxed{\pi_i^* = \psi_i^2}$
- Entropy rate: $s_{MERW} = s_{max} = \ln \lambda_{max}$

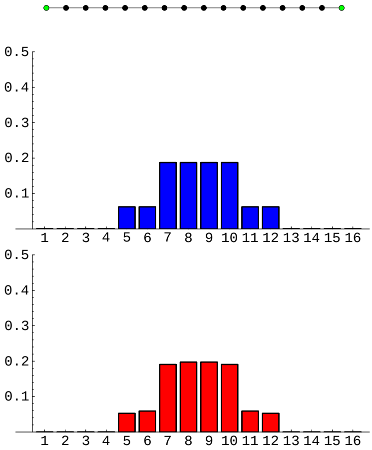
GRW vs. MERW on a finite chain



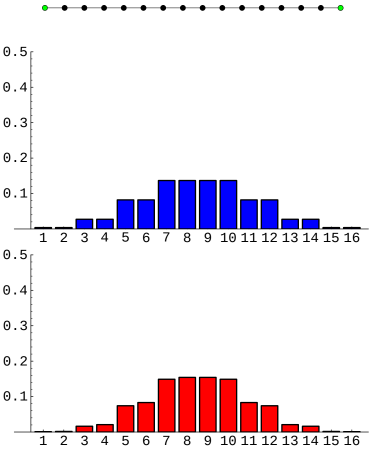
GRW vs. MERW on a finite chain



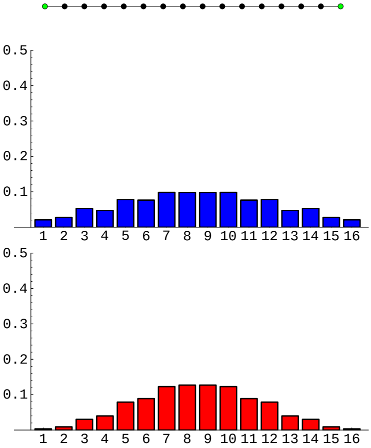
GRW vs. MERW on a finite chain



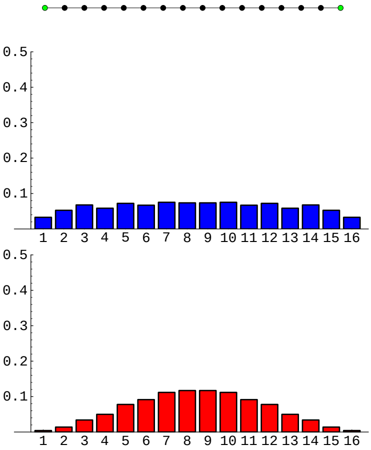
GRW vs. MERW on a finite chain



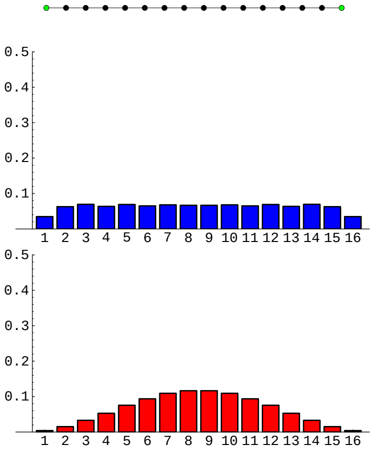
GRW vs. MERW on a finite chain



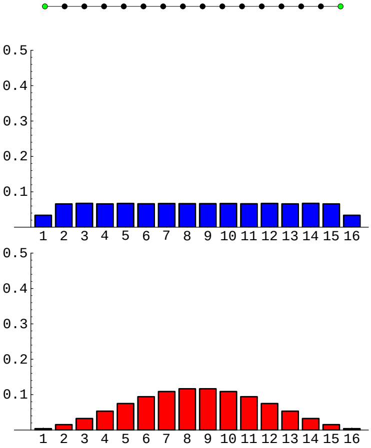
GRW vs. MERW on a finite chain



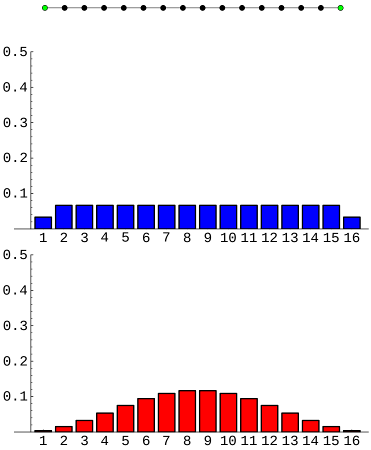
GRW vs. MERW on a finite chain



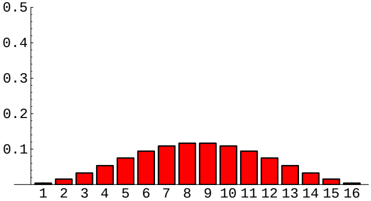
GRW vs. MERW on a finite chain



GRW vs. MERW on a finite chain

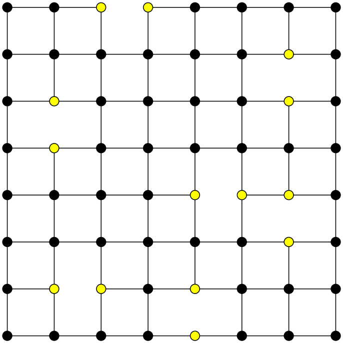


Defects repulsion



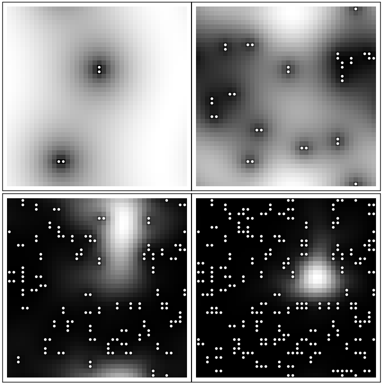
- $\pi_{*n} = \frac{2}{L+1} \sin^2 \frac{n\pi}{L+1}; \quad n = 1, \ldots, L$
- Repulsion at the endpoints (defects)

Lattice with dilution



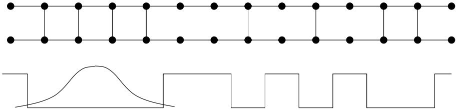
MERW on lattice with dilution

- 2d square lattice + a small fraction $q \ll 1$ of deleted links;
- Example: 40×40; $q = 0.001, 0.01, 0.05, 0.1$:



1d example + Lifshitz argument

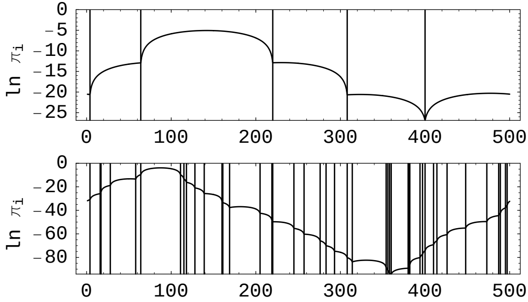
- ladder with randomly removed rungs:



- $\psi_{i+1} + \psi_{i-1} + r_i\psi_i = \lambda_{max}\psi_i, \;\; r_i = 1$ with prob. p ;
- Lifshitz, Nieuwenhuizen, Luck
- $-(\Delta\psi)_i + v_i\psi_i = E_0\psi_i; \quad v_i = 1 - r_i; \quad E_0 = k_{max} - \lambda_{max};$
- localization on the longest chain of rungs $l \sim \ln L/|\ln p|$
- $\psi_i \sim \sin i\pi/(l+1); \quad E_0 \sim \pi^2/l^2$

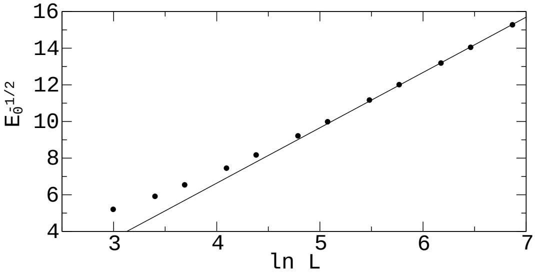
Numerical checks (1)

- $\pi_i^* = \psi_i^2$ for $L = 500$; $q = 1 - p = 0.01, 0.1$.



Numerical checks (2)

- $E_0 \approx (\pi |\ln p| / \ln L)^2$
- $L = 20, \ldots, 960;\ \ q = 1 - p = 0.1$



Lifshitz spheres in $D > 1$

- $-(\Delta\psi)_i + v_i\psi_i = E_0\psi_i$
- $-(\Delta\psi)_i = E_0\psi_i$ with Dirchlet boundary condition;
- in the largest spherical region free of defects;
- in 2D: radius: $R \approx (\ln L/(\pi|\ln p|))^{1/2}$

Summary

- GRW maximizes local entropy;
- MERW maximizes global entropy;
- localization in the presence of weak disorder of the lattice;
- classical localization (but it is mapped into quantum problem)
- localization is related to Lifshitz states of a random operator;
- Inequality:

$$\lambda_{max} \geq \exp\left(\frac{\sum_i k_i \ln k_i}{\sum_i k_i}\right) \quad \text{where} \quad k_i = \sum_j A_{ij}$$

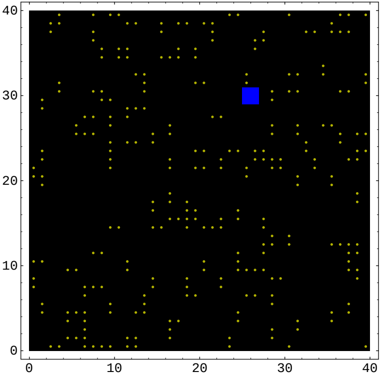
Open problems

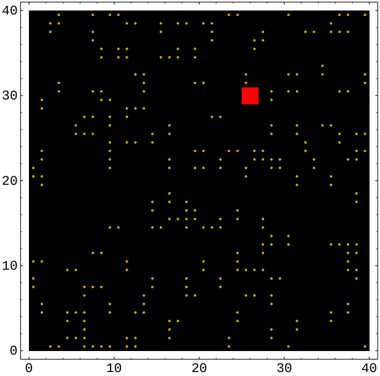
- Quantum amplitudes:

$$K_{ab} = \sum_{t} \sum_{\gamma_{ab}^{(t)}} \mathrm{e}^{-S_{\mathrm{E}}}; \quad S_{\mathrm{E}} \sim t$$

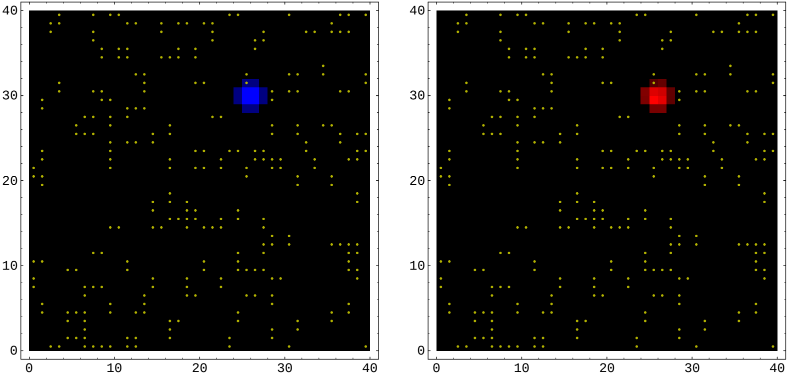
- Dynamics: entropy barriers of local Lifshitz spheres;

MERW dynamics

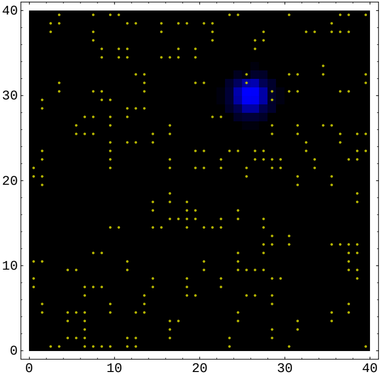


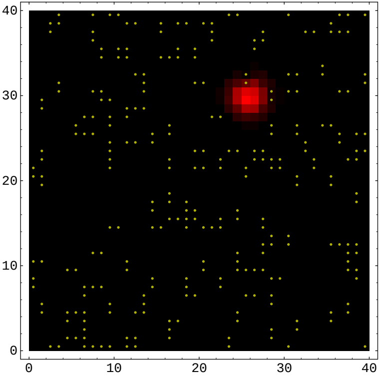


MERW dynamics

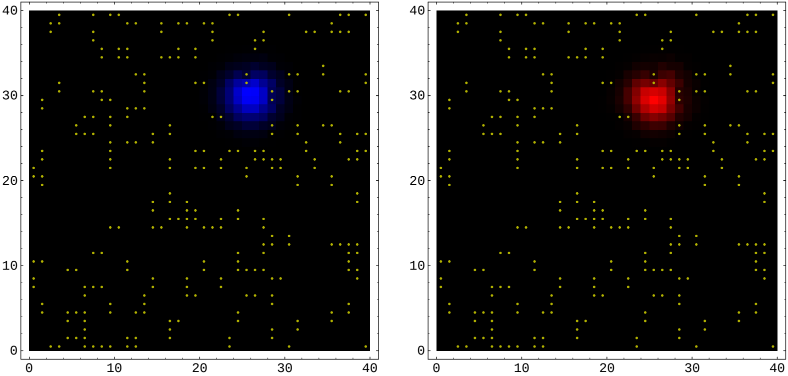


MERW dynamics

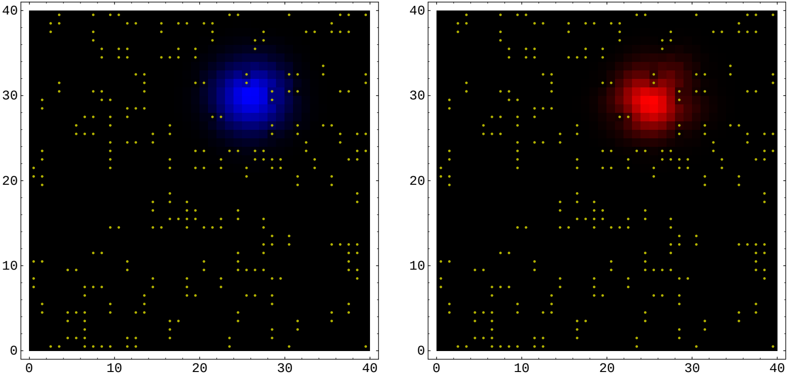




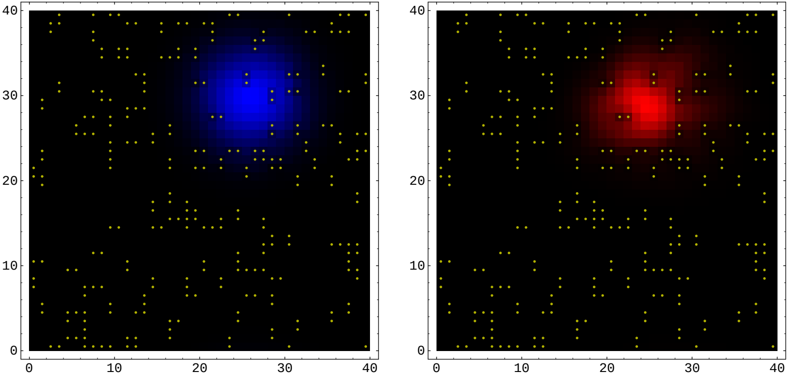
MERW dynamics



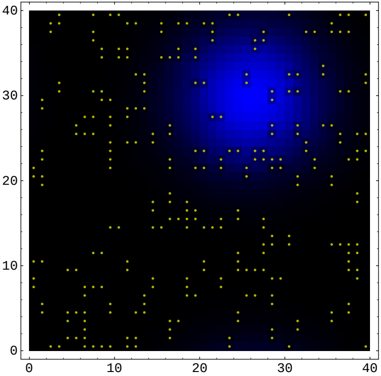
MERW dynamics

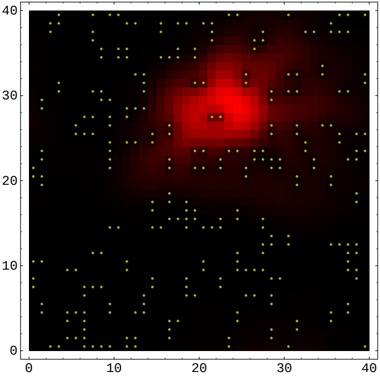


MERW dynamics

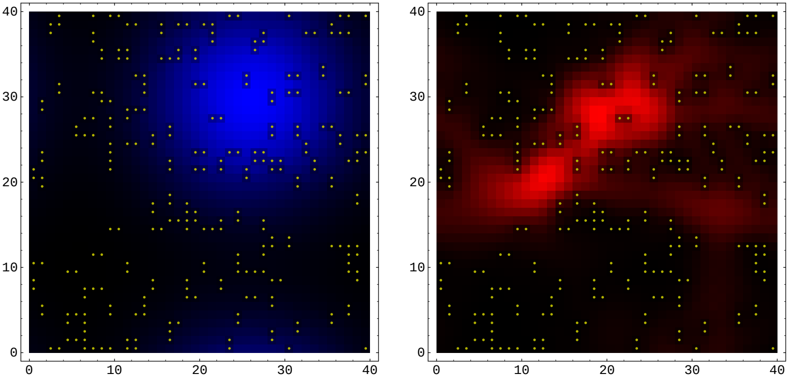


MERW dynamics

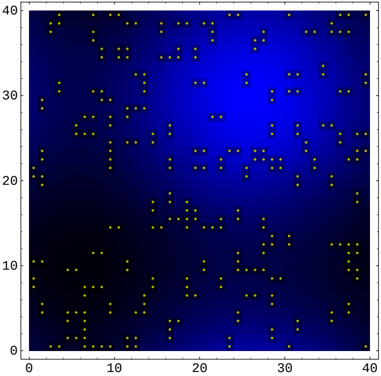


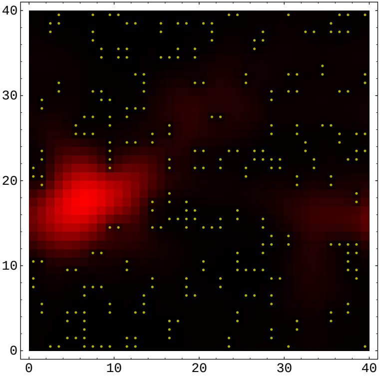


MERW dynamics

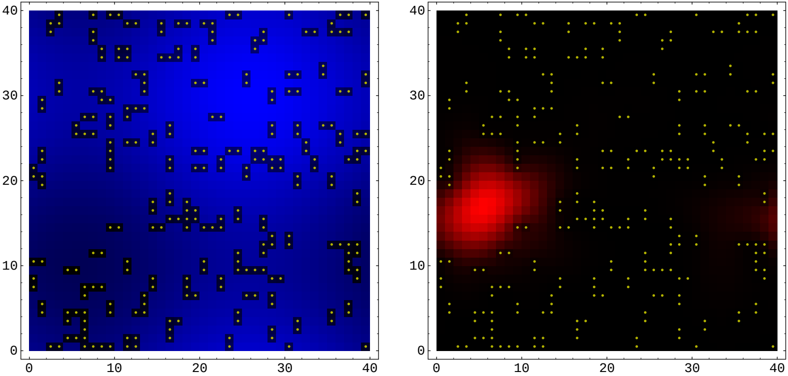


MERW dynamics





MERW dynamics



MERW dynamics