Multi-scale Modularity in Complex Networks

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Complexity & Networks

R. Lambiotte, J.-C. Delvenne and M. Barahona, *arXiv:0812:1770*
R. Lambiotte, Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt), 2010 Proceedings of the 8th International Symposium on, 546-553 (2010)
V.D. Blondel, J.-L. Guillaume, R. Lambiotte and E. Lefebvre, Fast unfolding of community hierarchies in large networks, J. Stat. Mech., (2008) P10008

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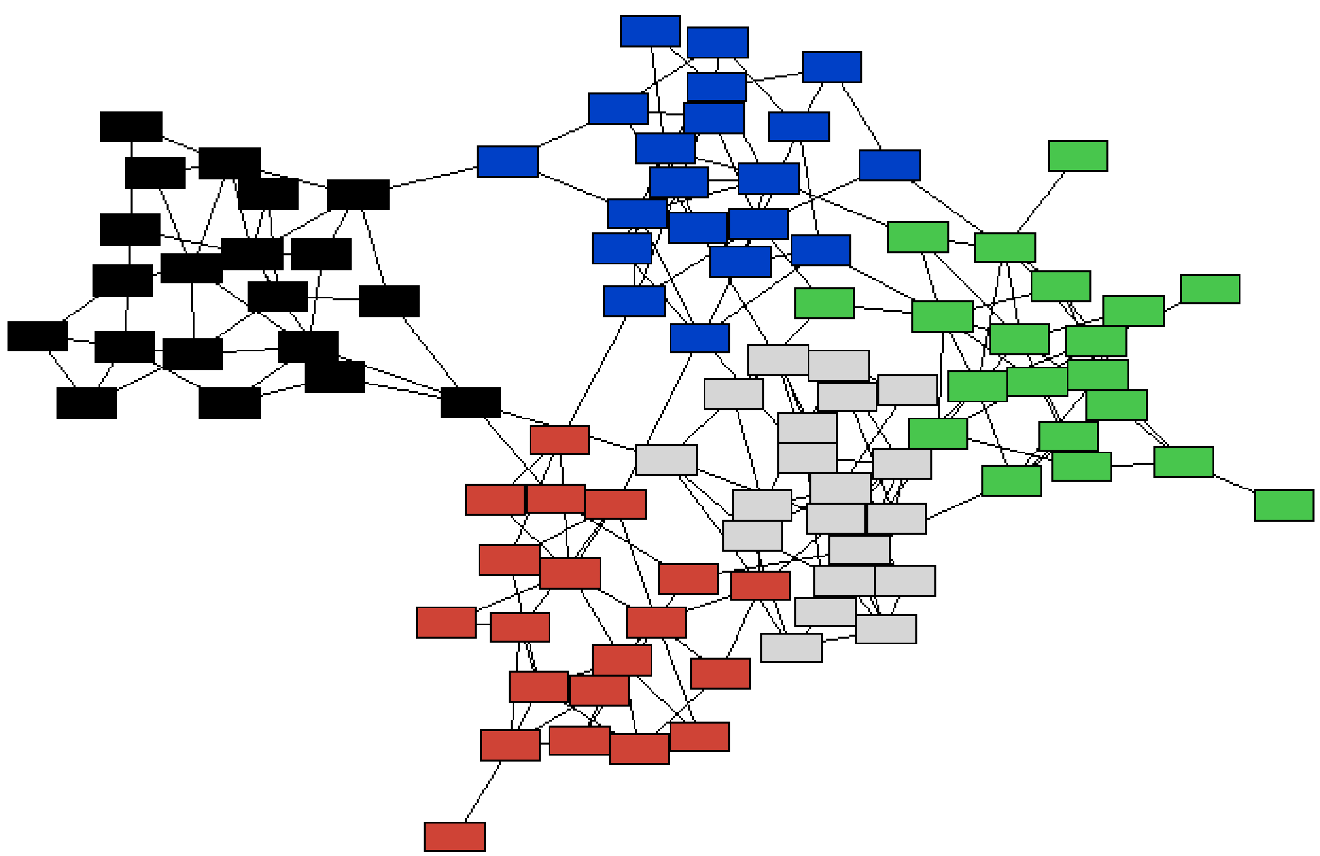
100 years of living science
100

1. Modules and Hierarchies
2. Stability of a partition
    a. Stability vs Modularity
    b. Time as a resolution parameter
    c. Different dynamics
3. Optimisation and selection of the most relevant time scales/robustness

Modular Networks

Most networks are very inhomogeneous and are made of modules:
many links within modules and a few links between different modules

Modules=communities



Internet

Power grids

Food webs

Metabolic networks

Social networks

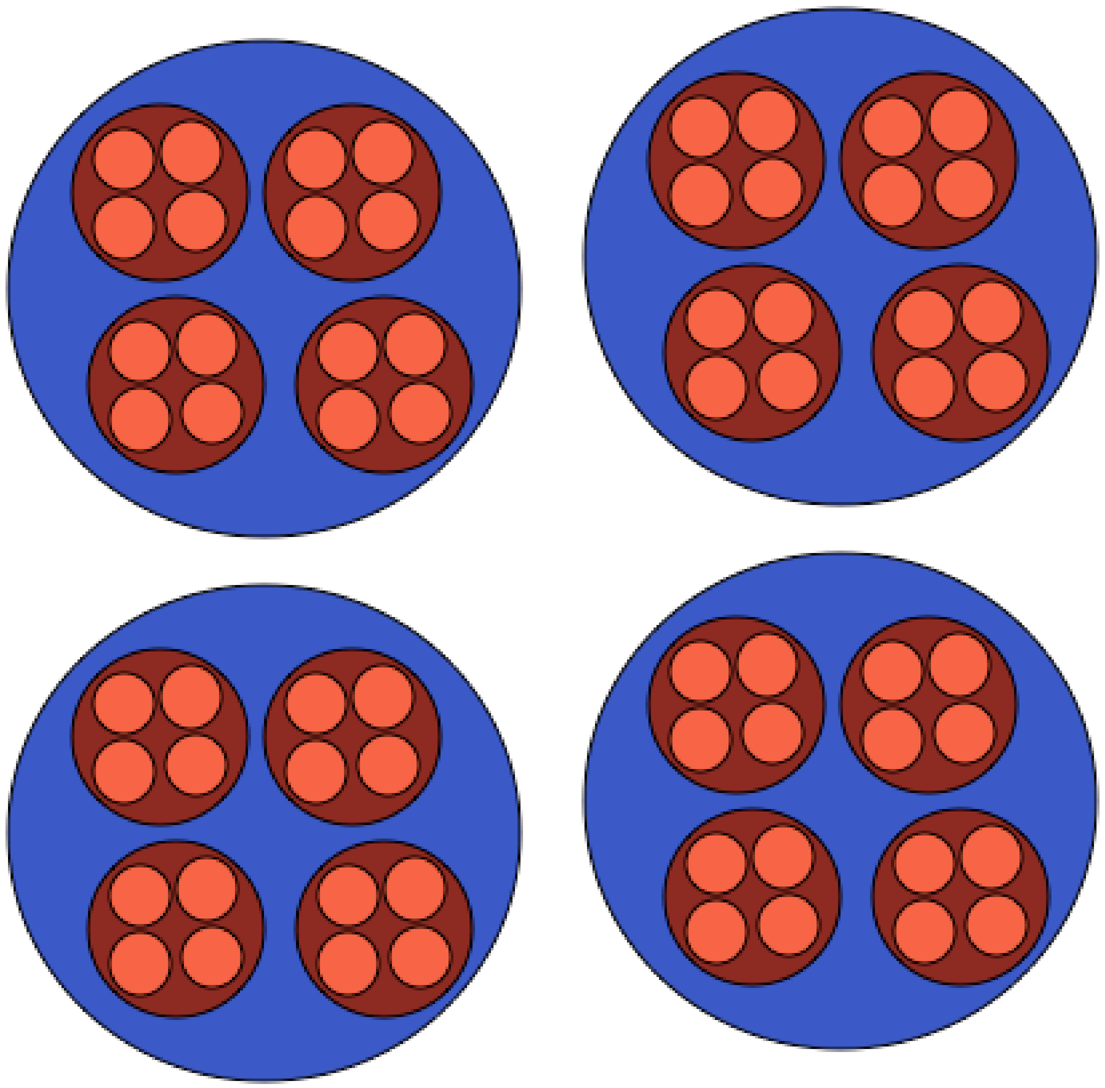
The brain

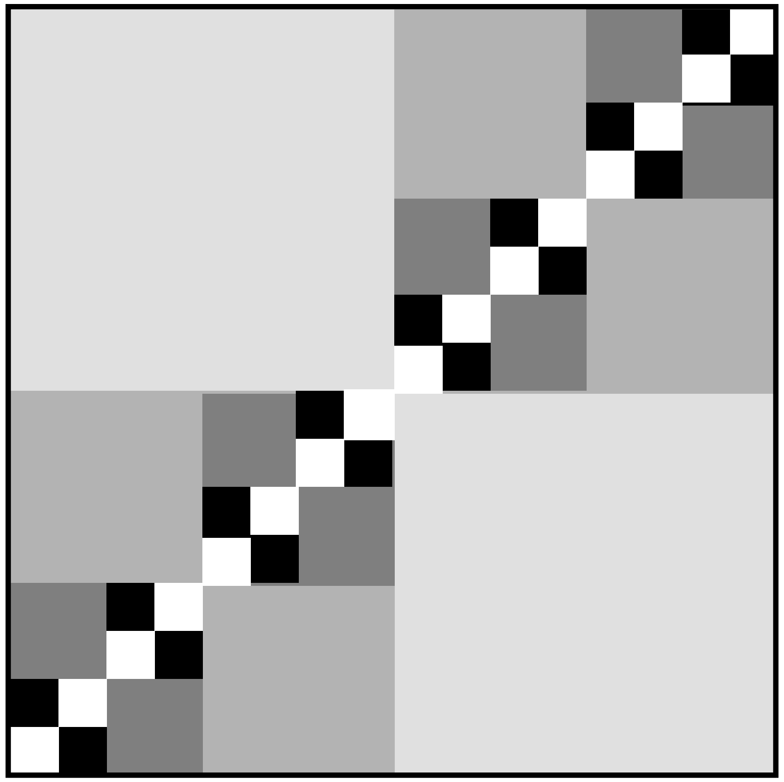
Etc.

Simon, H. (1962). The architecture of complexity. Proceedings of the American Philosophical Society, 106, 467–482.

Multi-scale Modular Networks

Networks have a hierarchical structure: modules within modules





Simon, H. (1962). The architecture of complexity. Proceedings of the American Philosophical Society, 106, 467–482.

Why communities, hierarchies?

Generic mechanism driving the emergence of modularity?

Why communities, hierarchies?

Generic mechanism**s** driving the emergence of modularity?

- Watchmaker metaphor: intermediate states/scales facilitate the emergence of complex organisation from elementary subsystems

- Separation of time scales: enhances diversity, locally synchronised states

- locally dense but globally sparse: advantages of dense structures while minimising the wiring cost

- in social systems, offer the right balance between dense networks (foster trust, facilitate diffusion of complex knowledge), and open networks (small diameter, ensures connectivity, facilitates diffusion of “simple” knowledge)

- naturally emerges from co-evolution and duplication processes

- Optimality of modular networks at performing tasks in a changing environment

- enhanced adaptivity and dynamical complexity, e.g. transient “chimera” states

- delivers highly adaptive processing systems and to solve the dynamical demands imposed by global integration and functional segregation (brain organisation)

D. Meunier, R. Lambiotte and E.T. Bullmore, “Modular and hierarchical organisation in complex brain networks”, Frontiers in NeuroScience (2010)

Modular Networks and dynamics

Is it possible to uncover those modules/hierarchies in large networks?

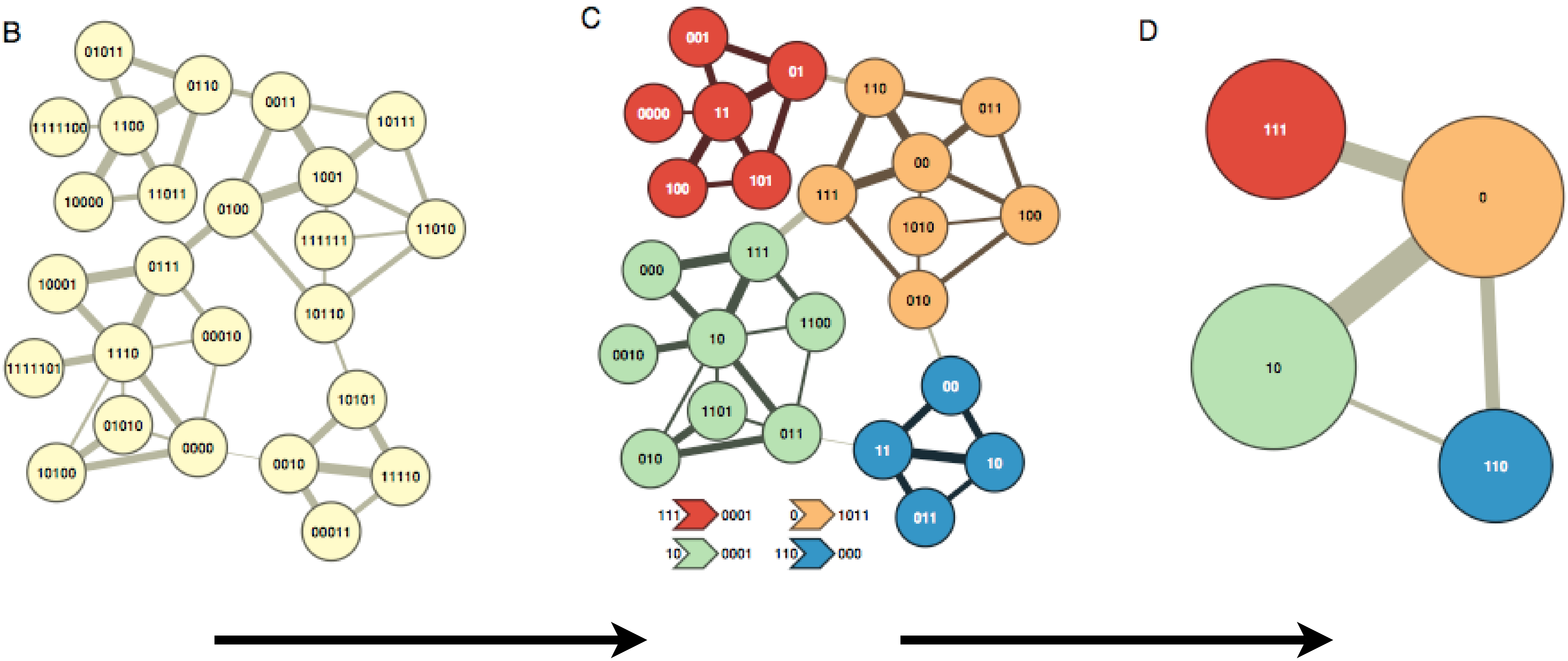
Hundreds of heuristics to optimise modularity.

How does such modularity affect dynamics?

A. Arenas, A. Diaz-Guilera and C.J. Pérez-Vicente (*Phys. Rev. Lett.*, 2006).

Modular Networks

Uncovering communities/modules helps to understand the structure of the network, to uncover similar nodes and to draw a readable map of the network (when N is large).



Find a partition of the network into communities

Coarse-grained description

Martin Rosvall and Carl T. Bergstrom, PNAS 105, 1118 –1123 (2008)

Modular Networks and dynamics

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Modular Networks and dynamics

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Hundreds of heuristics to optimise modularity.

Is it possible to use dynamics to characterize (and uncover?) the modular structure of a network?

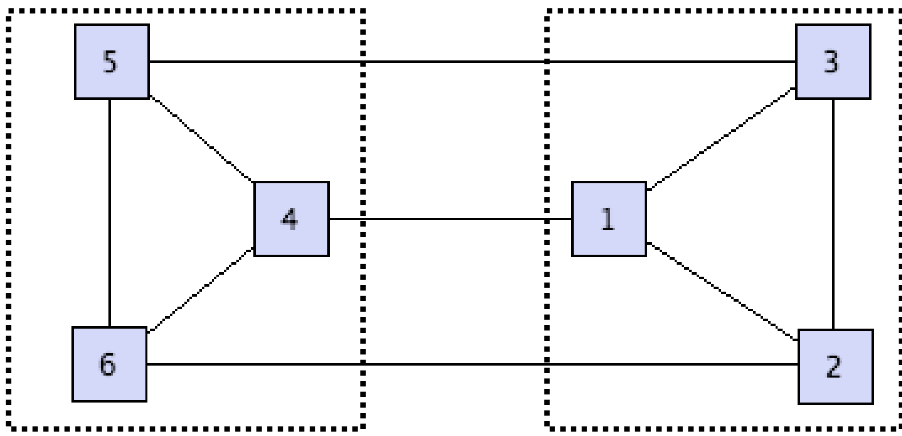
e.g. Walktrap (RW exploration), Rosvall and Bergstrom (PNAS, 2008), Fiedler (Laplacian spectrum)

How does such modularity affect dynamics?

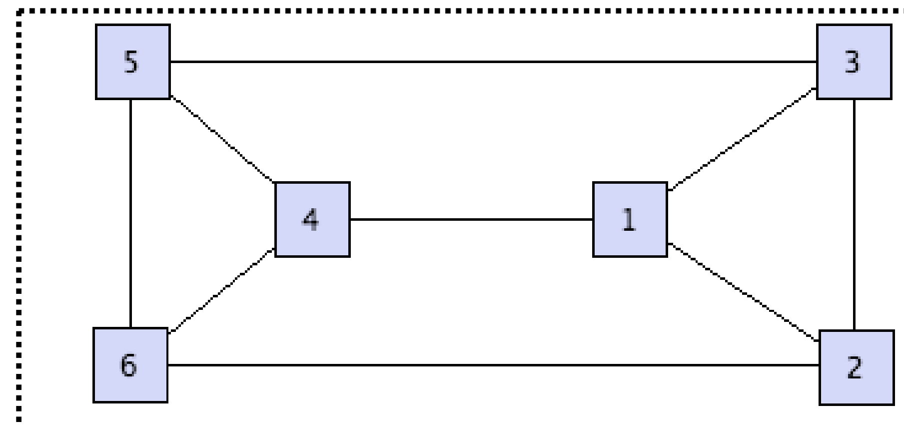
A. Arenas, A. Diaz-Guilera and C.J. Pérez-Vicente (*Phys. Rev. Lett.*, 2006).

Quality of a partition

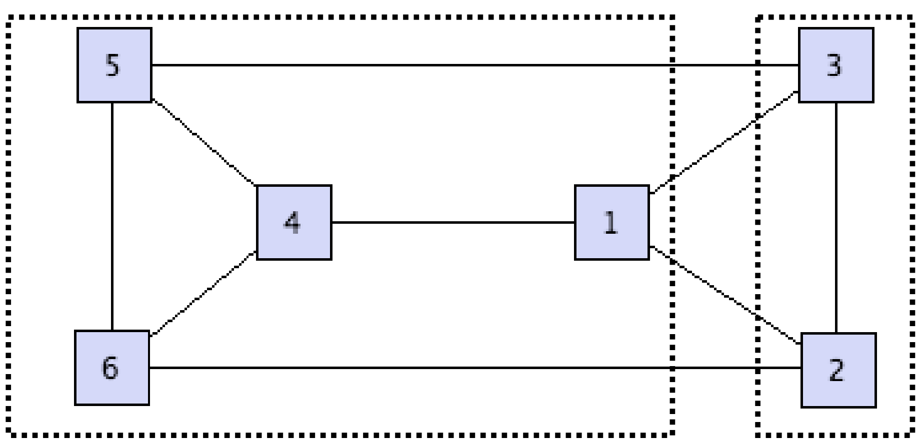
What is the best partition of a network into modules?



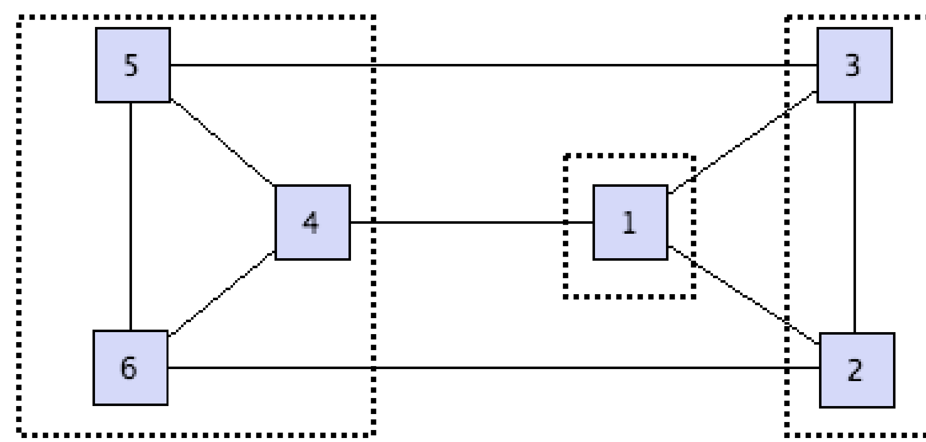
Q1



Q2



Q3



Q4

....

Modularity

Q = fraction of edges within communities - expected fraction of such edges

Let us attribute each node *i* to a community c_i

$$Q = \frac{1}{2m} \sum_{i,j} \Big[A_{ij} - P_{ij} \Big] \delta(c_i, c_j)$$

$$Q \in [-1, 1]$$

$$P_{ij} = \frac{k_i k_j}{2m}$$

expected number of links between i and j

$$\rightarrow \quad Q_C = \frac{1}{2m} \sum_{i,j} \Big[A_{ij} - k_i k_j / 2m \Big] \delta(c_i, c_j)$$

Modularity optimisation

Different types of algorithm for different applications:

Small networks (<10^2): Simulated Annealing

Intermediate size (10^2 - 10^4): Spectral methods, PL, etc.

Large size (>10^4): greedy algorithms, e.g. multiscale optimisation

	Karate	Arxiv	Internet	Web nd.edu	Phone	Web uk-2005	Web WebBase 2001
Nodes/links	34/77	9k/24k	70k/351k	325k/1M	2.04M/5.4M	39M/783M	118M/1B
CNM	.38/0s	.772/3.6s	.692/799s	.927/5034s	-/-	-/-	-/-
PL	.42/0s	.757/3.3s	.729/575s	.895/6666s	-/-	-/-	-/-
WT	.42/0s	.761/0.7s	.667/62s	.898/248s	.553/367s	-/-	-/-
Our algorithm	.42/0s	.813/0s	.781/1s	.935/3s	.76/44s	.979/738s	.984/152mn

V.D. Blondel, J.-L. Guillaume, RL and E. Lefebvre, *J. Stat. Mech.*, P10008 (2008).

Modularity

Resolution limit

Optimising modularity uncovers one partition

What about sub (or hyper)-communities in a hierarchical network?

Reichardt & Bornholdt

$$Q_\gamma = \frac{1}{2m} \sum_{i,j} \Big[A_{ij} - \gamma P_{ij} \Big] \delta(c_i, c_j)$$

Arenas et al.

$$Q(A_{ij} + r I_{ij})$$

Tuning parameters allow to uncover communities of different sizes

Reichardt & Bornholdt different of Arenas, except in the case of a regular graph where

$$\gamma = 1 + r / \langle k \rangle$$

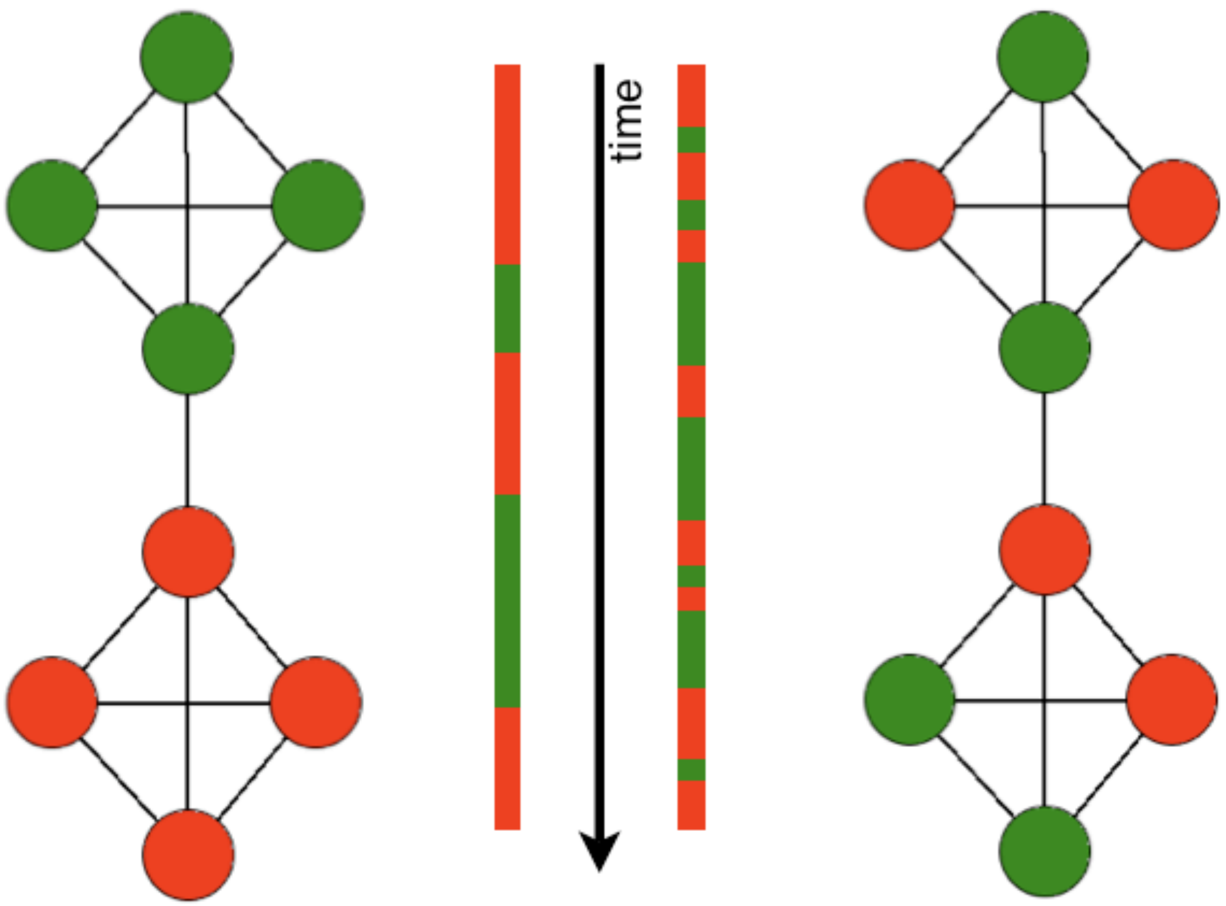
J. Reichardt and S. Bornholdt, Phys. Rev. E ***74****, 016110 (2006). Statistical mechanics of community detection*

A Arenas, A Fernandez, S Gomez, New J. Phys. ***10****, 053039 (2008). Analysis of the structure of complex networks at different resolution levels*

Stability

The quality of a partition is determined by the patterns of a flow within the network: a flow should be trapped for long time periods within a community before escaping it.

The stability of a partition is defined by the statistical properties of a random walker moving on the graph:



M. Rosvall and C. T. Bergstrom, PNAS 105, 1118 –1123 (2008)

J.-C. Delvenne, S. Yaliraki & M. Barahona, Stability of graph communities across time scales. arXiv:0812.1811.

Stability

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$$R(t) = \sum_{C \in \mathcal{P}} P(C, t_0, t_0 + t) - P(C, t_0, \infty)$$

$P(C, t_0, t_0 + t)$ probability for a walker to be in the same community at times t_0 and $t_0 + t$ when the system is at equilibrium

$P(C, t_0, \infty)$ probability for two independent walkers to be in C (ergodicity)

J.-C. Delvenne, S. Yaliraki & M. Barahona, Stability of graph communities across time scales. arXiv:0812.1811.

a. Modularity vs Stability

Let us consider a random walk on an undirected network:

$$p_{i;n+1} = \sum_j \frac{A_{ij}}{k_j} p_{j;n} \quad \xrightarrow{\text{equilibrium}} \quad p_i^* = k_i/2m$$

$$R(1) = \sum_{i,j} \left[\frac{A_{ij}}{k_j} \frac{k_j}{2m} - \frac{k_i k_j}{(2m)^2} \right] \delta(c_i, c_j)$$

Probability that a walker is in the same community initially and at time t=1

Same probability for independent walkers

$$R(1) = Q = \frac{1}{2m} \sum_{i,j} \left[A_{ij} - \frac{k_i k_j}{2m} \right] \delta(c_i, c_j)$$

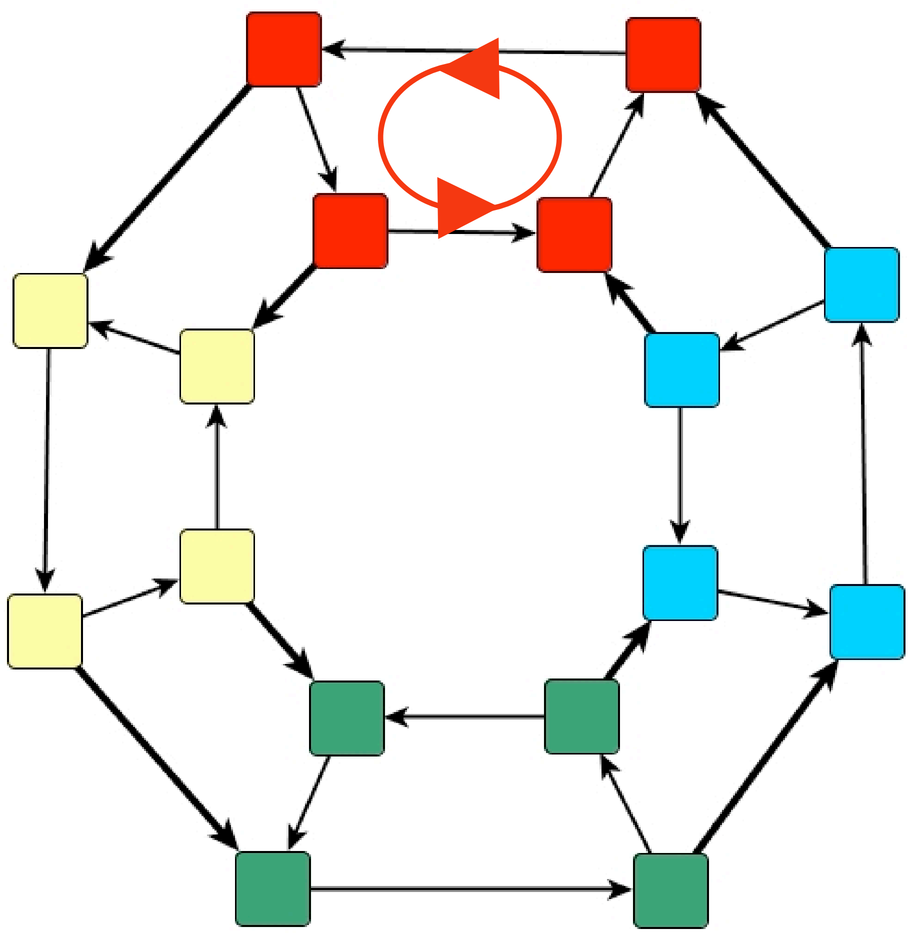
a. Modularity vs Stability

Let us consider a random walk on a directed network:

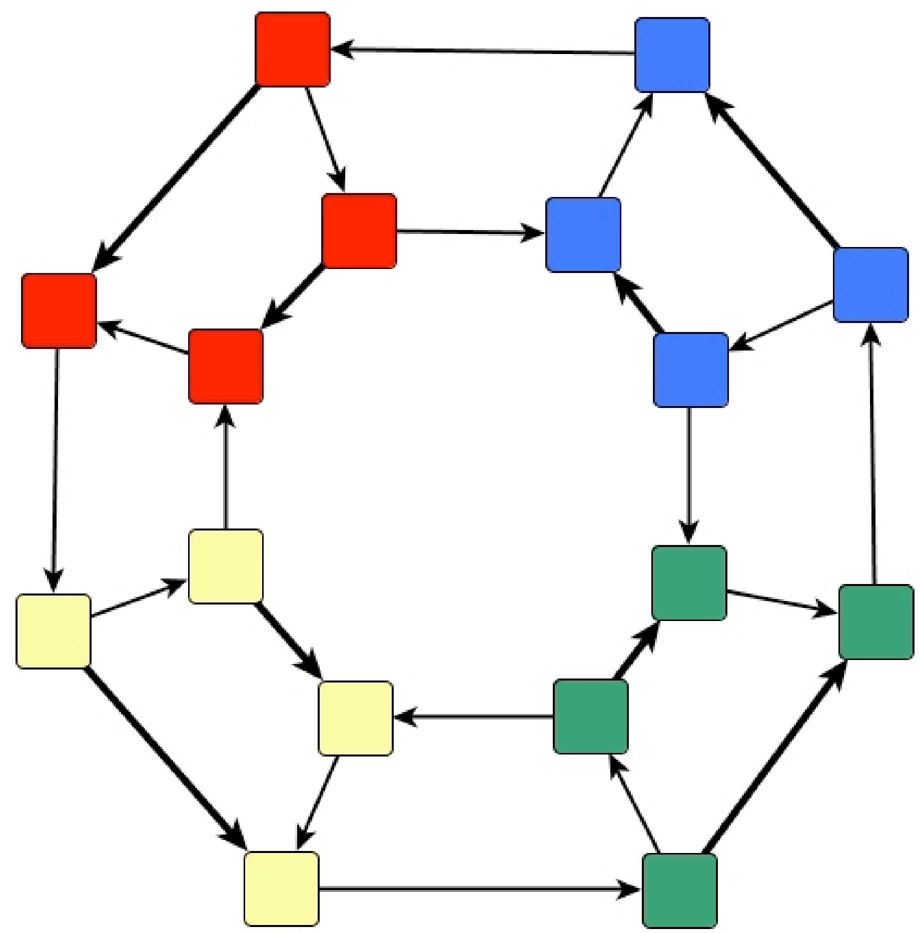
$$p_{i;n+1} = \sum_j \frac{A_{ij}}{k_j^{out}} p_{j;n} \quad \xrightarrow{\text{equilibrium}} \quad p_i^* = \pi_i$$

$$R(1) = \sum_{i,j} \left[\frac{A_{ij}}{k_j^{out}} \pi_j - \pi_i \pi_j \right] \delta(c_i, c_j) \neq Q$$

a. Modularity vs Stability



Flow-based modules



Combinatorial modules

M. Rosvall and C. T. Bergstrom, PNAS 105, 1118 –1123 (2008)

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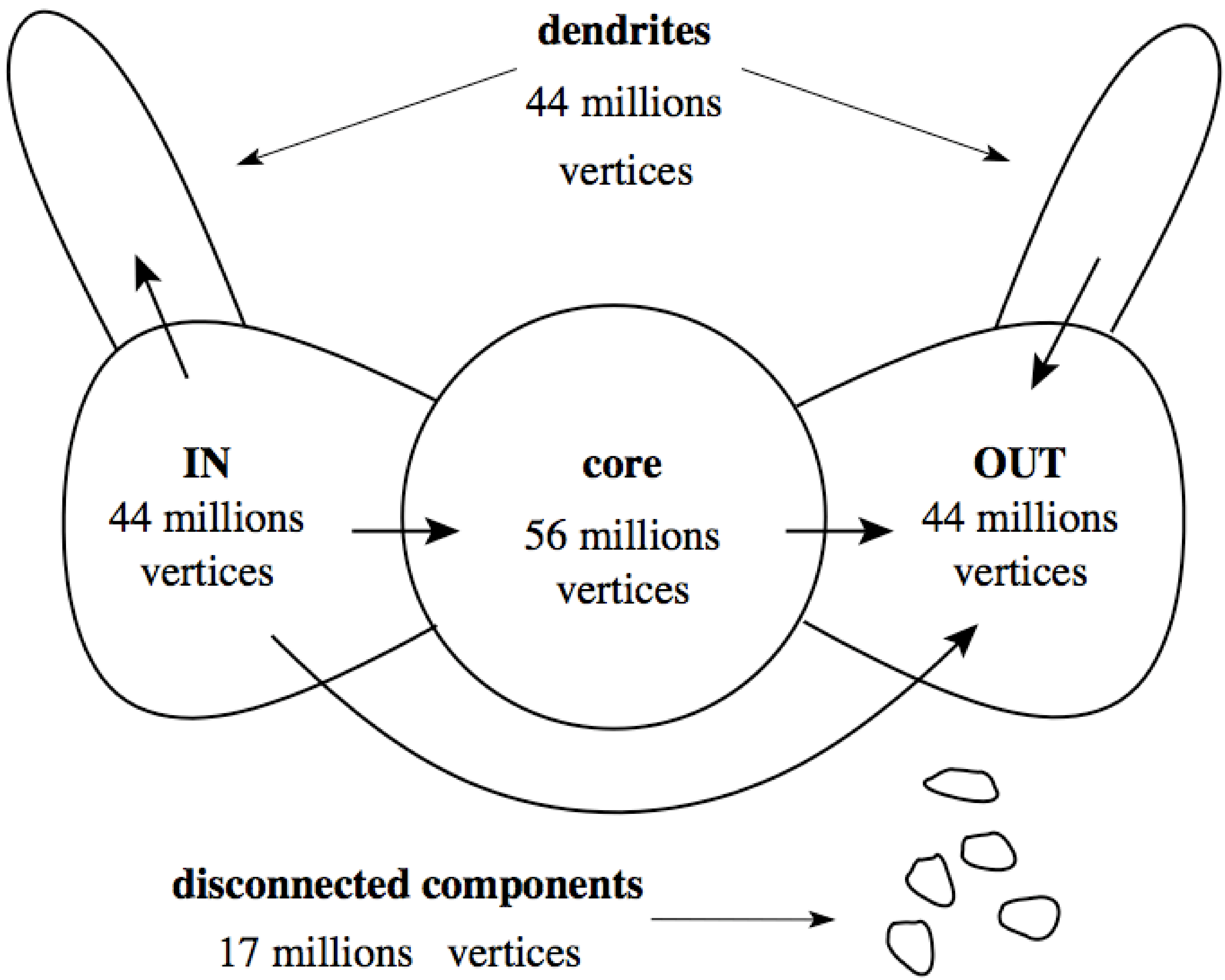

J.-L. Guillaume and M. Latapy, Proceedings of ALGOTEL'02

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$$R(1) = \sum_{i,j} \left[\frac{A_{ij}}{k_j^{out}} \pi_j - \pi_i \pi_j \right] \delta(c_i, c_j) \neq Q$$

$$R(1) \neq Q(A) \quad \text{but} \quad R(1) = Q(Y)$$

$$Y = \frac{X + X^T}{2} \qquad X_{ij} = \frac{A_{ij}}{k_j^{out}} \pi_j$$

b. Stability: time as a resolution parameter

Let us consider a continuous-time random walk with Poisson waiting times

$$\dot{p}_i = \sum_j \frac{A_{ij}}{k_j} p_j - p_i \quad \xrightarrow{\text{equilibrium}} \quad p_i^* = k_i/2m$$

$$R(t) = \sum_{i,j} \left[\left(e^{t(B-I)}\right)_{ij} \frac{k_j}{2m} - \frac{k_i k_j}{(2m)^2} \right] \delta(c_i, c_j)$$

$$B_{ij} = A_{ij}/k_j$$

Probability that a walker is in the same community initially and at time t

Same probability for independent walkers

$$L_{ij} = A_{ij}/k_j - \delta_{ij}$$

Spectral decomposition: $$\frac{1}{2m} \sum_C \sum_{i,j \in C} \sum_{\alpha=2}^{N} e^{t\lambda_\alpha} v_{\alpha;i} v_{\alpha;j}$$

b. Stability: time as a resolution parameter

What are the optimal partitions of R_t?

t=0 $$R(0) = 1 - \sum_{i,j} \frac{k_i k_j}{(2m)^2} \delta(c_i, c_j)$$ → Communities=single nodes

t small $$R(t) \approx (1-t)R(0) + tQ_C \equiv Q(t)$$

favours single nodes — modularity

!! Q_t equivalent to the Hamiltonian formulation of Reichardt and Bornholdt (t=1/gamma)

....

When t goes to infinity, the optimal partition is made of 2 communities (by spectral decomposition, i.e. dominant eigenvector of the Laplacian)

c. Other dynamics: other modules

$$R(t) = \sum_{C \in \mathcal{P}} P(C, t_0, t_0 + t) - P(C, t_0, \infty)$$

Stability depends on partition, time and on the details of the RW process

There are infinitely many ways to define a RW on the same graph: what is the waiting distribution between jumps (passage to continuity) and is there a bias?

$$T_{ij}^{(\alpha)} = \frac{\alpha_i A_{ij}}{\sum_k \alpha_k A_{kj}}$$

If flows are important, one should chose a model as close as possible to the dynamics actually taking place on the system.

PS: biased RWs are unbiased RWs on another graph defined by

$$A'_{ij} = \alpha_i A_{ij} \alpha_j$$

In practice: Optimisation

The stability *R(t)* of the partition of a graph with adjacency matrix *A* is equivalent to the modularity *Q* of a time-dependent graph with adjacency matrix *X(t)*

$$X_{ij}(t) = \left(e^{t(B-I)}\right)_{ij} k_j \qquad X_{ij}(t) = X_{ji}(t)$$

which is the flux of probability between 2 nodes at equilibrium and whose generalised degree is

$$\sum_j X_{ij}(t) = k_i$$

$$R(t) = \sum_{i,j} X_{ij}(t)/2m - k_i k_j/(2m)^2 \ \delta(c_i, c_j) = Q(X(t))$$

For very large networks: $R(t) \approx (1-t)R(0) + tQ_C \equiv Q(t)$

In practice: Selection of the most relevant scales

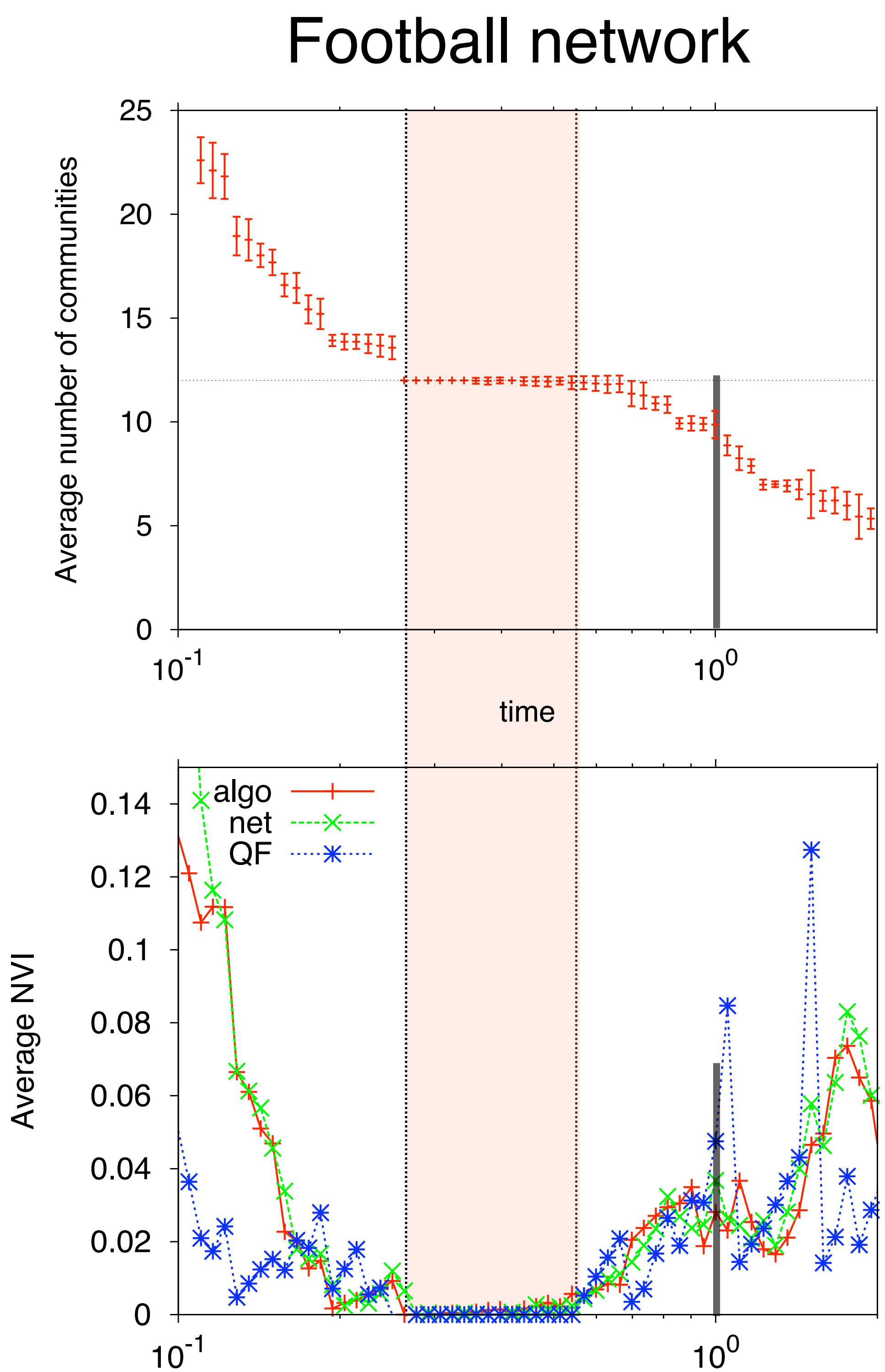
The optimization of R(t) over a period of time leads to a sequence of partitions that are optimal at different time scales.
How to select the most relevant scales of description?

The significance of a particular scale is usually associated to a certain notion of the robustness of the optimal partition. Here, robustness indicates that a small modification of the optimization algorithm, of the network, or of the quality function does not alter this partition.

We look for regions of time where the optimal partitions are very similar. The similarity between two partition is measured by the normalised variation of information.

D.J. Fenn, M.A. Porter, M. McDonald, S. Williams, N.F. Johnson, N.S. Jones, *Chaos* 19, 033119 (2009).
M. Meilă, *J. Multivariate Anal.* 98, 873 (2007).

Verification on empirical networks and multi-scale benchmarks

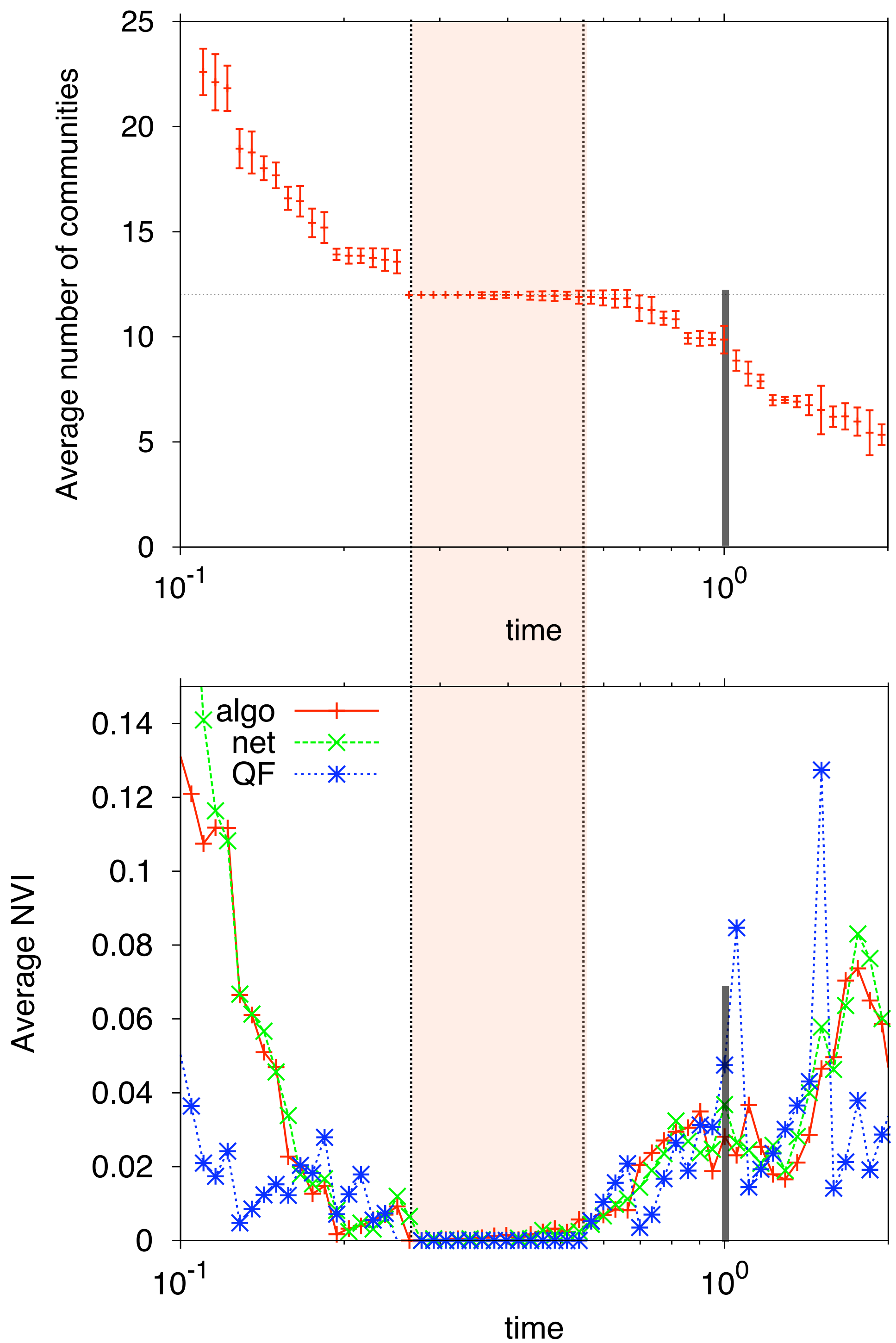


algo: for each t, 100 optimizations of Louvain algorithm while changing the ordering of the nodes

$$\langle V \rangle_{\text{algo}}(t) = \frac{2}{T(T-1)} \sum_{i=1}^{T} \sum_{i'=i+1}^{T} \hat{V}(\mathcal{P}_i(t), \mathcal{P}_{i'}(t)).$$

Verification on empirical networks and multi-scale benchmarks

Football network



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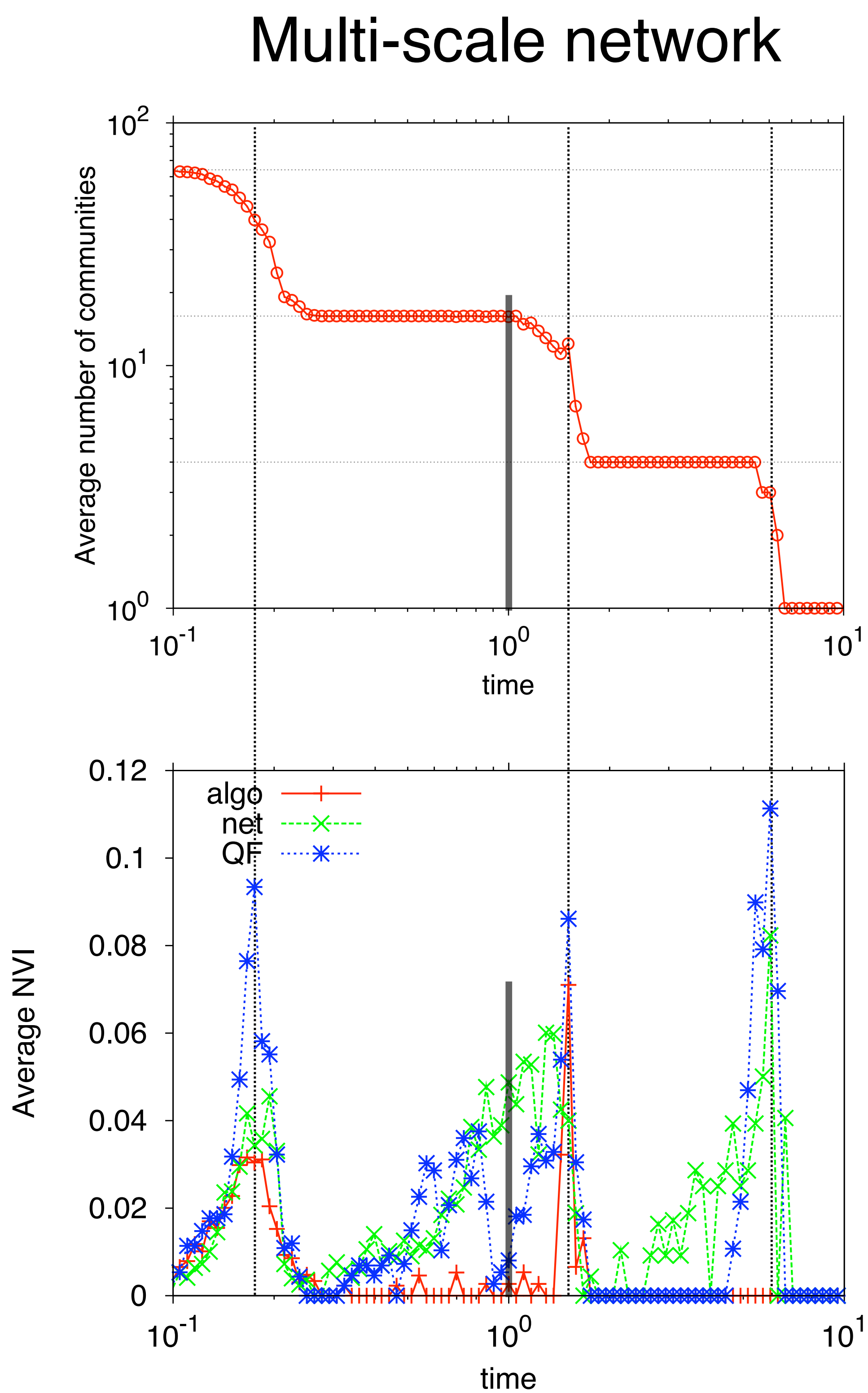
net: for each t, 100 optimizations with a fixed algorithm but randomized modifications of the network

QF: for each t, one optimization. Partitions at 5 successive values of t are compared.

Compatible notions of robustness

Lack of robustness => high degeneracy of the QF landscape: uncovered partitions are not to be trusted; wrong resolution

Verification on empirical networks and multi-scale benchmarks



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Conclusion

- Relation between dynamics and the hierarchical structure of networks
- Dynamical formulation for the quality of a partition
- Modularity and Stability are radically different in the case of directed networks
- Changing time allows to zoom in and out
- Different dynamics, different modules
- Notions of robustness to uncover significant scales

Original Louvain method to optimise modularity available on *http://findcommunities.googlepages.com*

Generalized codes to optimise Q_t available on http://*www.lambiotte.be*

Thanks to J.-L. Guillaume (for providing his c++ code)

R. Lambiotte, J.-C. Delvenne and M. Barahona, *arXiv:0812:1770*

R. Lambiotte, *arXiv:1004.4268*

V.D. Blondel, J.-L. Guillaume, R. Lambiotte and E. Lefebvre, J. Stat. Mech., P10008 (2008).

T. Evans and R. Lambiotte, *Phys. Rev. E*, **80** (2009) 016105

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D. Meunier (Cambridge)
V. Blondel (Louvain)

Hierarchical modularity in human brain functional networks, D. Meunier, R. Lambiotte, A. Fornito, K.D. Ersche, E.T. Bullmore, *Frontiers in NeuroInformatics* ***3****, 37 (2009)*