Complex Networks

Frank Schweitzer

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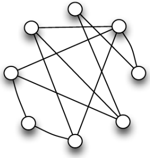
Outline

1. What are complex networks?
2. Growth of complex networks
3. Dynamics on complex networks

What are complex networks?



- **What are networks?**
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  - `mathematics:` *graph* consists of *vertices* and *edges*

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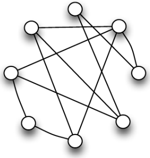

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- **What is 'complex'?**
  - *'complex'* does not mean *'complicated'*
  - complex systems: regularities, universality on the systemic level
  - *complexity as a system property* ⇒ not to simplify, not to reduce

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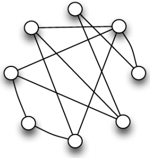

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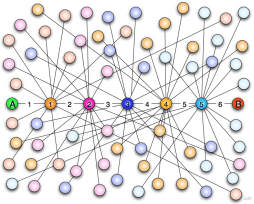
- **complex networks**
  - *representation* of complex systems
  - system elements ⇒ *nodes*
  - interactions ⇒ *links*



What are complex systems?

- system comprised of a *large* number of *strongly* interacting (similar) subsystems (entities, processes, or '*agents*')
  - examples: brain, insect societies (ants, bees, termites), ...

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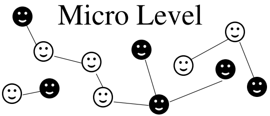

$\Longleftrightarrow$

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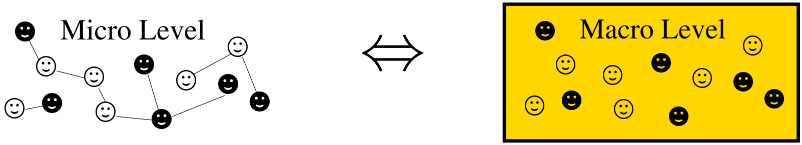
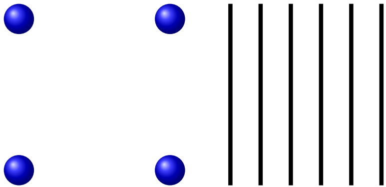


- **challenge:** The micro-macro link
  - How are the properties of the elements and their interactions ("microscopic" level) related to the dynamics and the properties of the whole system ("macroscopic" level)?

How to describe complex networks?

- **complex network:** agents ⇒ *nodes*, interactions ⇒ *links*



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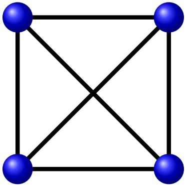

fully connected network

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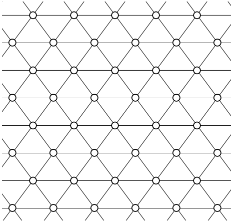

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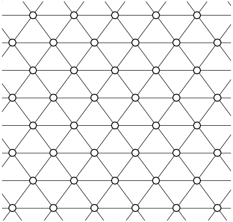

- **degree distribution** $P(k)$
  - count the number of links k_i of each node i, do a histogram
    - ⋆ $z = K/N$: *average degree*

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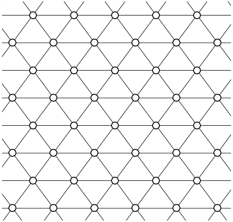

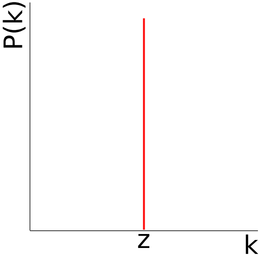

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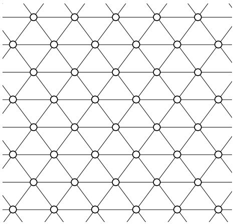
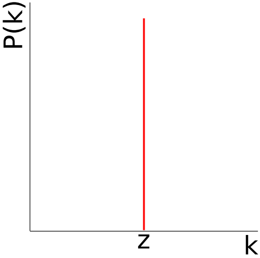
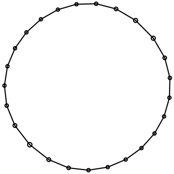




- **degree distribution** $P(k)$
  - count the number of links k_i of each node i, do a histogram
    - $z = K/N$: *average degree*
  - delta-function ⇒ *regular lattice* ⇒ **complete order**
    - regular network as a limiting case of complex network
    - of interest: fully connected networks ⇔ mean-field approximation

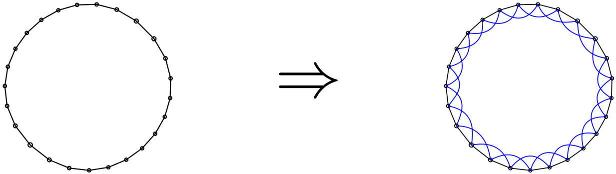
Topologies

- **chain with $z = 2$**



Topologies

- **chain with** $z = 4$: interaction with second-nearest neighbors



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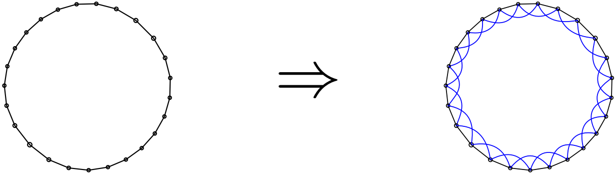
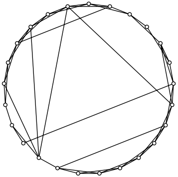
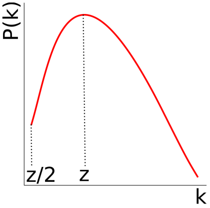


- **rewire links** with probability p

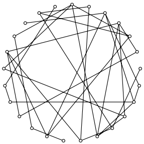


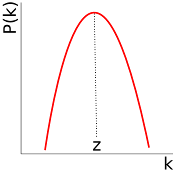


  - ▸ **small-world network:** low-dimensional regular lattice + randomness

Topologies: More heterogeneity

- **random network**: (rewiring probability $p \to 1$)
  - $P(k)$: Poissonian distribution around average degree z (Erdös-Reyni)





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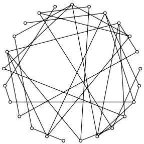
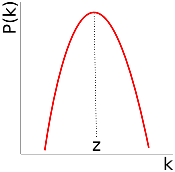
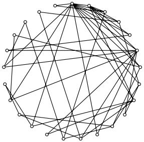
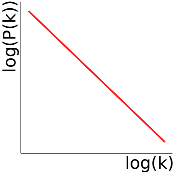



- **scale-free network**: existence of a few *hubs*
  - $P(k) \propto k^{-\alpha}$: Power law distribution, no average degree z defined

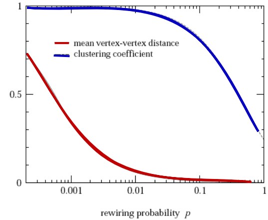



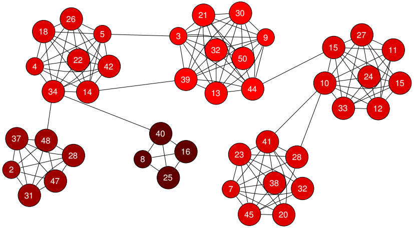
Topologies: Conclusions

- **distinct topological features**
  - **ordered** vs. **small-world** vs. **random** vs. **scale-free** networks
  - *heterogeneity:* nodes have a (very!) different numbers of neighbors
  - *complexity* ranges between order (*lattice*) and randomness (ER)

Topologies: Conclusions

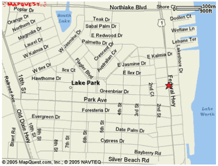
- **distinct topological features**
  - **ordered** vs. **small-world** vs. **random** vs. **scale-free** networks
  - *heterogeneity:* nodes have a (very!) different numbers of neighbors
  - *complexity* ranges between order (*lattice*) and randomness (ER)
- **consequences**
  - *average distance* between any two nodes varies
    clustering varies: how many of my neighbors are also neighbors
    - ★ high for regular network, low for small world network
    - ★ consequencies for e.g. information transport, infection

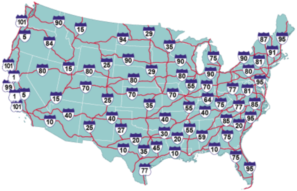




Real-world examples of complex networks

- **regular networks**:e.g. urban roads, highway systems in rual areas





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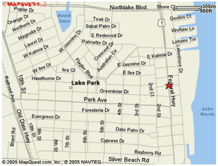
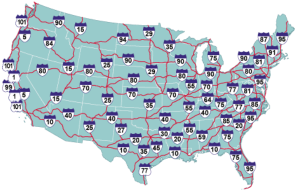
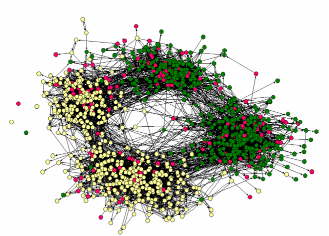




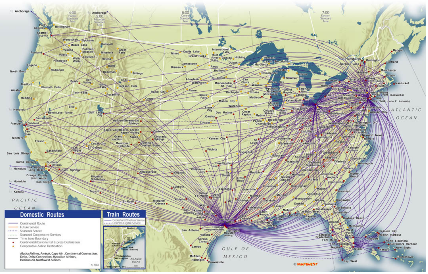
- **Small-world networks**: e.g. friendship networks

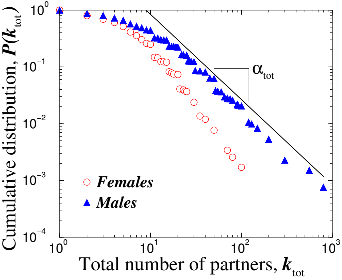


color: different races (Yellow - White Race, Green - Black Race, Pink - Other
top/bottom different ages (middle and high school)
J. Moody, ASJ 107 (2001) 679-716

Examples: Scale-free networks

- World Wide Web, Wikipedia, aviation network, sexual contacts





Weighted networks

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  - interaction with different weights, time dependence, ...

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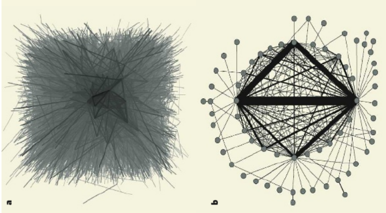
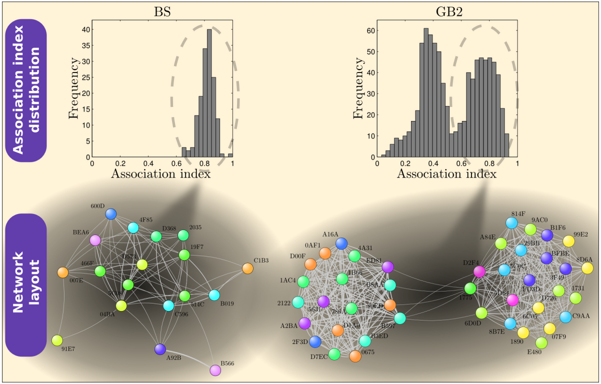
  - interaction with different weights, time dependence, ...
- **example: Fedwire interbank payment network**
  - links represent transaction volumes
  - **existence of a backbone**: involves small number of nodes



(K. Soramäki *et al.* Physica A **379** (2007) 317-333)
(left) Thousands of banks and tens of thousands of links representing USD 1.2 $\times 10^{12}$ in daily transactions; **(right)** Core of the network: 66 banks accounting for 75 % of transfers, 25 banks being completely connected.

Example: Community structure in Bats



- association: measures the time indiviuals spend together
- larger colony splits into communities ⇒ *social units*

G. Kerth, N. Perony, F.S. (2010, submitted)

Directed networks

- **links have a meaning**
  - asymmetry of interaction ⇒ direction
- **example: international trade network** (ITN)
  - dominant flow patterns, node importance (centrality) ⇒ integration

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- **example: international trade network** (ITN)
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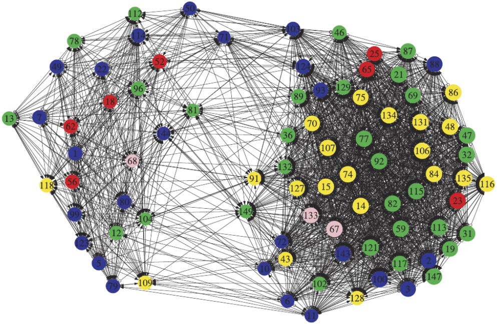
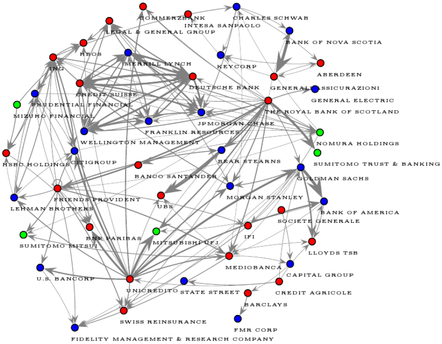
- **example: Italian overnight money market** (*Caldarelli*, ...)
  - relation between lenders/borrowers, response to exogeneous factors



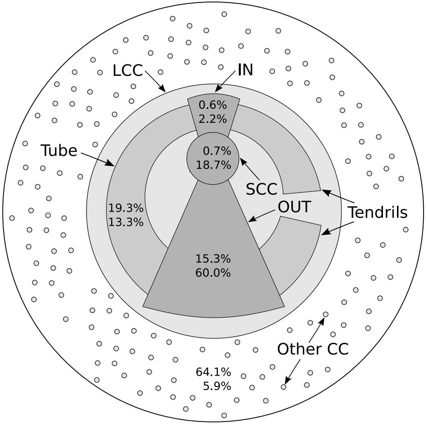
Example: Ownership in transnational companies

- **directed network** of ownership ⇒ *control*



Example: International financial network
Nodes represent *major* financial institutions, links the strongest existing relations, node colors different geographical areas

Example: Network of Transnational Companies (TNCs)

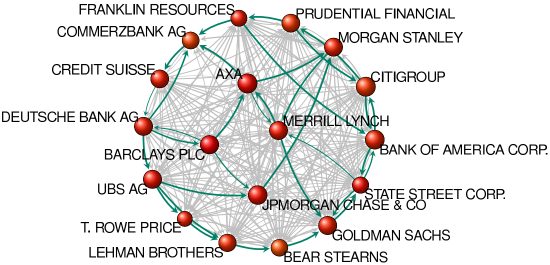


Size of components scaled by (log) number of TNC.

- Largest connected component (LCC) contains giant bow-tie:
  - IN-section, strongly connected component (SCC) core, OUT-section,
  - tubes and tendrils.
- Remaining small connected components (CC).
- Numbers refer to
  - percentage of contained TNC,
  - total TNC operating revenue.

S. Vitali, J. Glattfelder, S. Battiston (ETH Zurich)

Problem: Self-Ownership



Excerpt of the network of financial intermediaries in the SCC

- 75% of the ownership of the SCC firms stays within the SCC
  - propagation of financial distress increases systemic risk
  - cross-ownership decreases competition

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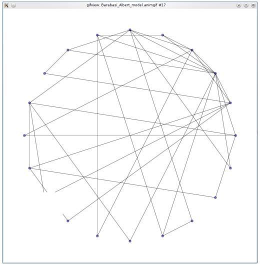
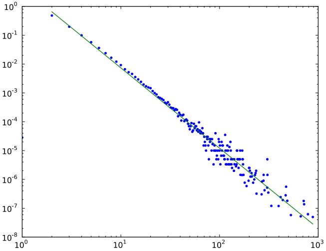
How do complex network grow?

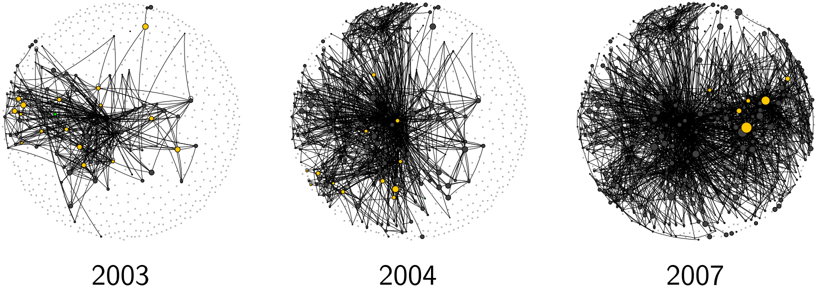
- **preferential attachment**
  - at each time step, add a new node i with m=const. links
  - connect i with other nodes j with probability $p_j = k_j / \sum_l k_l$
    - ★ the more links j has, the more it will get *
  - **result**: emergence of hubs ⇒ *scale-free network* $P(k) \propto k^{-\gamma}$ $(\gamma = 3)$





* *"Law of proportionate growth"* (Gibrat, 1931)

Example: Growth of Open Source Software

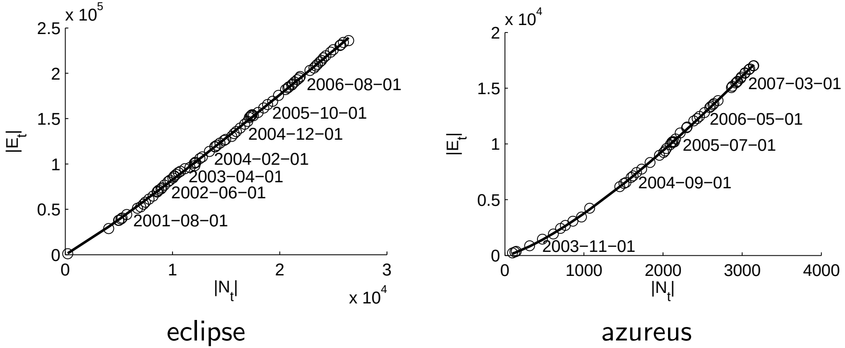


Class network of JUNG, a framework for network visualization

- *nodes:* represent Java files (classes)
- *links:* represent dependencies ⇒ references to other classes
  - **Data source:** 19 Java projects
  - monthly snapshots of dependency network and CVS logs
  - final sizes: from 1.856 to 28.898 nodes
- **What are the laws of network growth?**

Empirics: Accelerated Growth

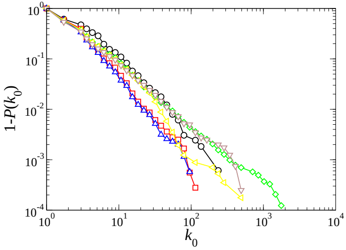
Growth of total number of links according to $K(N) \propto N^{\beta}$

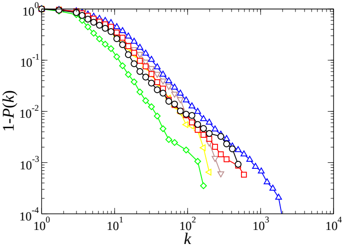


- $\beta \geq 1$: *increasing density of network*
  - confirmed for 'small' network sizes → saturation? $[\beta(t) \to 1]$

Empirics: Initial/Final Degree Distributions

- (left) **intitial degree**: $n(k_0) \propto k_0^{-\alpha}$
  - highly heterogeneous (compare standard BA: $k = const.$)
- (right) *final degree*: $n(k) \propto k^{-\gamma}$





Data: six OSS projects: `AspectJ` (black), `Azureus` (red), `Eclipse` (green), `Jedit` (blue), `Jena` (yellow), `Yale` (brown)

*C.J. Tessone, M.M. Geipel, F.S., PRL (subm.)

A Formal Approach to Network Dynamics

- **Aim**: **Relation between α, β, γ**
  - identify universal scaling laws for k, K, N
  - **more importantly:** *link different dimension of OSS*
    - *software design* (initial conditions $\Rightarrow \alpha$)
    - *developer activities* (growth dynamics $\Rightarrow \beta$)
    - *structure of final product* (link dependencies $\Rightarrow \gamma$)

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- **Assumption 1: nodes have an initial degree k_0**
  - nodes are added with fixed rate: $N \leftrightarrow t$
  - new node is linked to k_0 other nodes
  - k_0 randomly drawn from $g(k) = (\alpha - 1)k^{-\alpha}$
- **Assumption 2: preferential attachment**
  - new nodes link to highly connected nodes more frequently

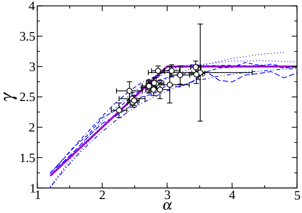
Results and Comparison with OSS

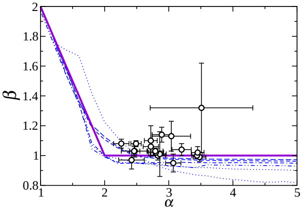
- **result for total number of links** $K(t) \sim t^{\beta}$

$$\beta = \begin{cases} 3-\alpha & \text{if } \alpha < 2 \quad \Rightarrow \text{accelerated growth} \\ 1 & \text{if } \alpha \geq 2 \quad \Rightarrow \text{linear growth} \end{cases}$$

- **result for final degree distribution** $n(k) \propto k^{-\gamma}$

$$\gamma = \begin{cases} \alpha & \text{if } \alpha < 3 \quad \Rightarrow \textit{initial degree distribution} \text{ dominates} \\ 3 & \text{if } \alpha \geq 3 \quad \Rightarrow \textit{preferential attachment} \text{ dominates} \end{cases}$$





*C.J. Tessone, M.M. Geipel, F.S., PRL (subm.)

Outline

1 What are complex networks?

2 Growth of complex networks

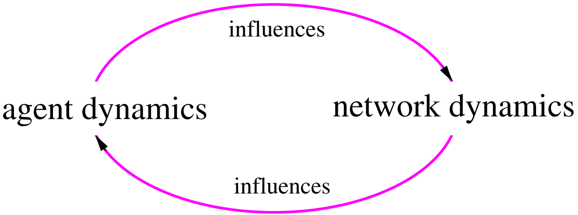
3 Dynamics on complex networks

What is missing?

- **So far: the links**
  - *topologies of networks*: structure and dynamics
  - ‘the art of drawing lines between nodes’

What is missing?

- **So far: the links**
  - *topologies of networks*: structure and dynamics
  - 'the art of drawing lines between nodes'
- **what about the nodes?**
  - agents have their own *internal dynamics*
  - strategically decide about *link formation* (humans, firms, ...)
- **feedback between nodes and links**



Example: Convergence toward shared behavior

- agent i: social behavior $x_i(t) \in [0, ..., 1]$
  - *utility* from social interaction with agents j:

$$\text{utility}_i(t) = \sum_j \text{benefits}_{ij}(t) - \text{costs}_{ij}(t)$$

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1. *assumption:* utility increases if everyone shares same behavior
   - benefit: $b = \text{const.}$, costs: $\sim \Delta x$

$$u_i(t) = \sum_j b - c\,|x_i - x_j|$$

2. *assumption:* interaction ij occurs only iff $u_{ij}(t) > u_{\text{thr}}$

$$|x_i - x_j| < \varepsilon = (b - u_{\text{thr}})/c$$

   - possibility of interaction depends on *'open-mindedness'* ε
3. *assumption:* interaction leads to more similar behavior of i and j

$$x_i(t+1) = x_i(t) + \mu\,[x_j(t) - x_i(t)]$$

   - $\mu = 0.5$: both agents adopt the 'mean' behavior

Influence of emerging in-groups

- interacting agents added to each other's in-group I_i and I_j
  - partnership relations from past interactions
  - *effective behaviour* x_i^{eff} considers *mean in-group behaviour* $\bar{x}_i^I$

    $$x_i^{\mathrm{eff}} = (1-\alpha_i)x_i + \alpha_i \bar{x}_i^I$$

  - group influence α_i increases with group size
- permanent influence of in-group on interaction: $\left| x_i^{\mathrm{eff}} - x_j^{\mathrm{eff}} \right| < \varepsilon$
  - search for new partners is costly $\rightarrow$ keep past partners
  - keep behavior close to past partners to allow further interaction

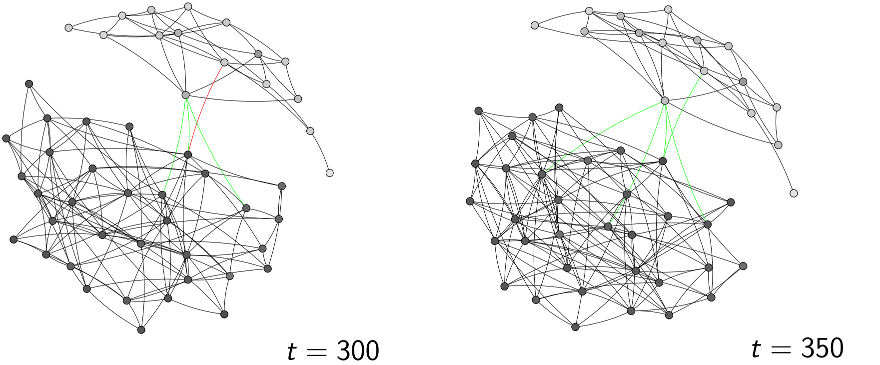
Co-evolution of social network and behavior

- randomly choose agents i, j at time t

1. **link dynamics** (considers existing in-group)
   - $\Delta x^{\mathrm{eff}}(t) < \varepsilon \Rightarrow$ link formation (interaction)
   - $\Delta x^{\mathrm{eff}}(t) > \varepsilon \Rightarrow$ no link created or *existing link is removed*
2. **dynamics in individual behavior** (considers $x_i(t)$, $x_j(t)$)
   - interacting agents become more similar
3. **adjustment of effective behavior**
   - agent i, j: $x_i \to x_i^{\mathrm{eff}}$, $x_j \to x_j^{\mathrm{eff}}$
   - in-groups of i and j: x_i^{eff}, x_j^{eff} affected by changed $\bar{x}^{I_i(t)}$, $\bar{x}^{I_j(t)}$

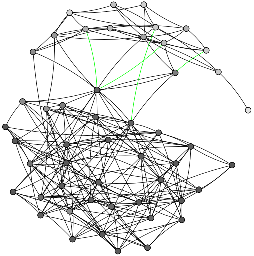
Result: *feedback between agents' behavior and their in-group structure* $\Rightarrow$ Computer simulation

Group Influence: two nearly separated components...

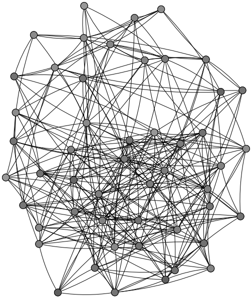


- 50 agents, $\varepsilon = 0.3$
  - green link: agents would not interact without group influence
  - red link: agents would not interact anymore

… finally united



$t = 400$



$t = 500$

- group influence (on average and a large range of ε)
  - fosters coalescence of components
  - increases maximum component size
- $\Rightarrow$ consensus toward a common behavior

Systemic Risk

- risk that whole system (of many interacting agents) **fails**
  - *financial sector* (banks, companies),
  - *power grids* (blackout due to overload)
  - *material science* (bundles of fibers)
- **common features**
  - failure of few agents is amplified ⇒ system failure
  - individual agent dynamics: fragility, threshold for failure
  - interaction: network topology

Micro Dynamics: Individual Agent

- **node** i with interaction matrix $\mathbf{A}$
  - *state* $s_i(t) \in \{0, 1\}$: 'healthy', 'failed'
  - *fragility* $\phi_i(t) > 0$: susceptibility to fail, may depend on other nodes
  - (individual) *threshold* θ_i for failure
- key variable: *net fragility*:

$$z_i(t) = \phi_i(t, \mathbf{s}, \mathbf{A}) - \theta_i$$

- *deterministic dynamics*

$$s_i(t+1) = \Theta[z_i(t)]$$

  - $s_i = 1$ if $z_i(t) \geq 0$; $s_i = 0$ if $z_i(t) < 0$
- **global fraction of failed nodes** $\Rightarrow$ *prediction*

$$X(t) = \frac{1}{n} \sum_{i=1}^{n} s_i(t)$$

  - **systemic risk:** $X(t \to \infty) = X^\star \to 1$

Models with constant load

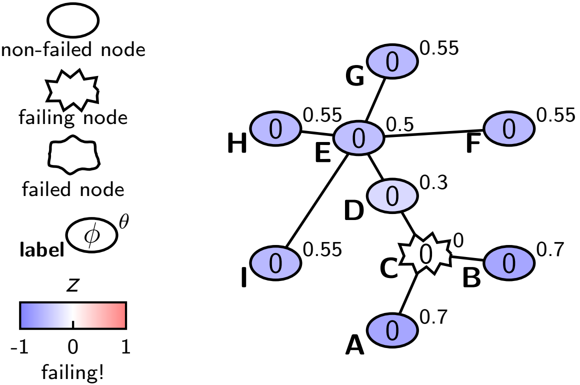
- fragility ϕ_i of agent *i* depends on failure of neigbors, s_j
- **(i) 'inward' variant**: increase of fragility depends on *in-degree*

$$\phi_i(t) = \frac{1}{k_i^{\mathrm{in}}} \sum_{j \in \mathrm{nb}_{\mathrm{in}}(i,A)} s_j(t)$$

- **(ii) 'outward variant':** increase of fragility depends on *out-degree*
  - load of failing node (i.e. 1) is shared equally among neighbors

$$\phi_i(t) = \sum_{j \in \mathrm{nb}_{\mathrm{in}}(i,A)} \frac{s_j(t)}{k_j^{\mathrm{out}}}$$

Example: Inward variant - node C fails



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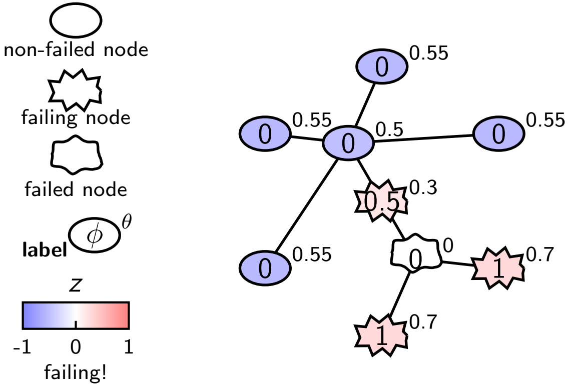

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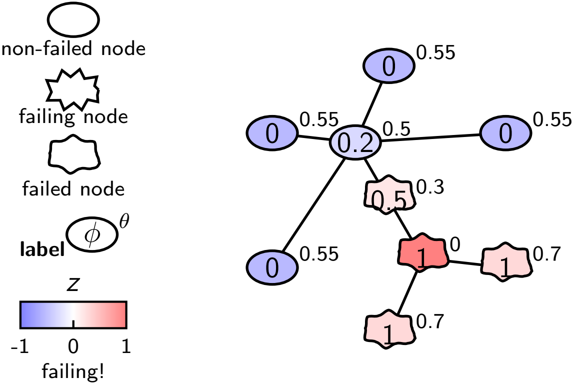
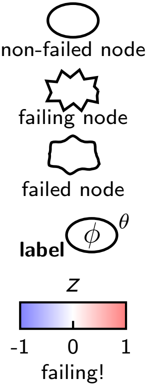
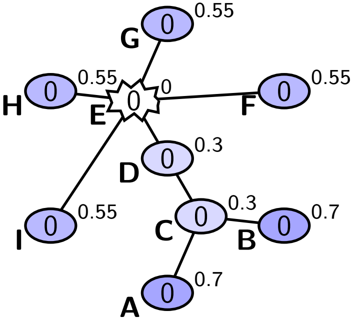


- low degree node ⇒ high vulnerability to fail
  - failure causes little damage, cascade stops after 2 steps ⇒ no 'systemic risk'

Example: Inward variant - node E fails





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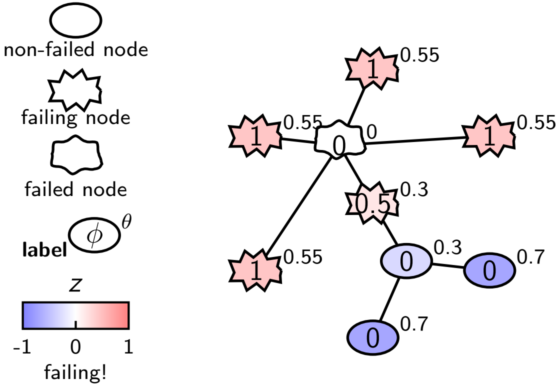

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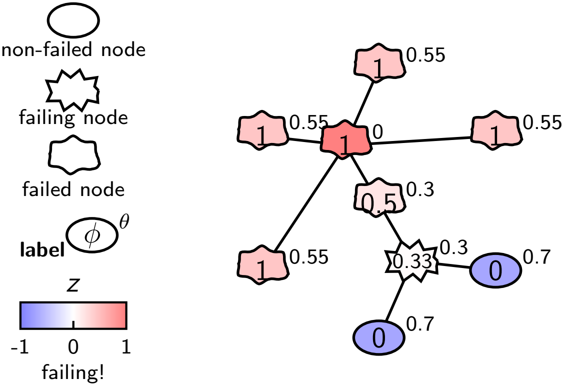

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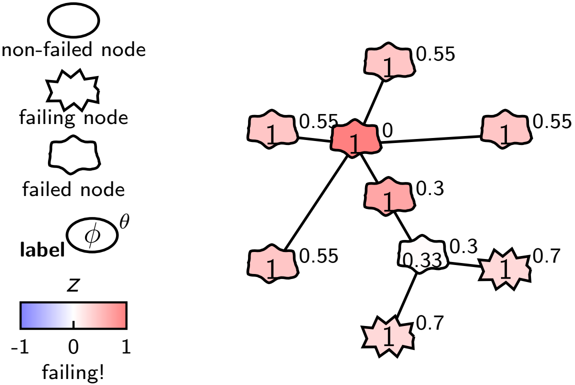

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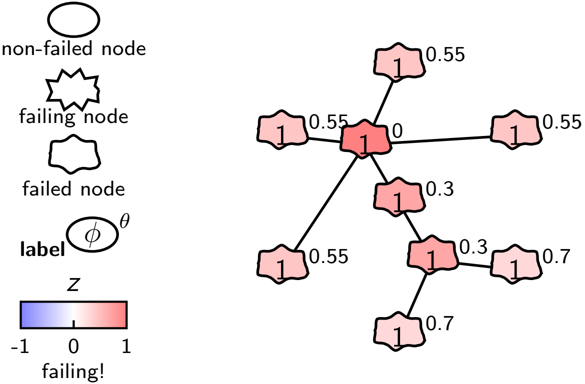
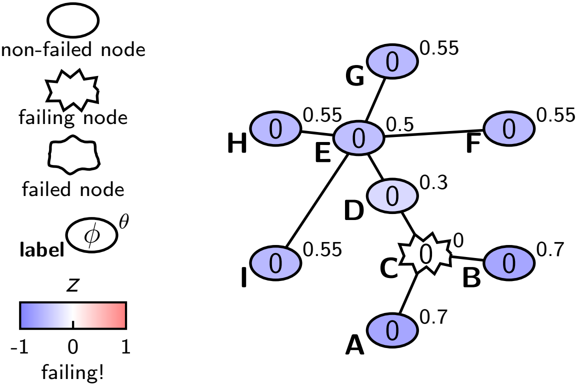


- high degree node ⇒ low vulnerability to fail
  - failure causes big damage (to low degree nodes), cascade involves all nodes ⇒ 'systemic risk'

Example: Outward variant - node C fails



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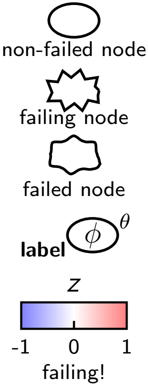

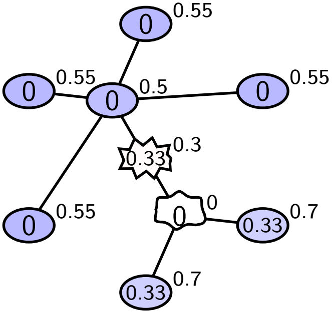

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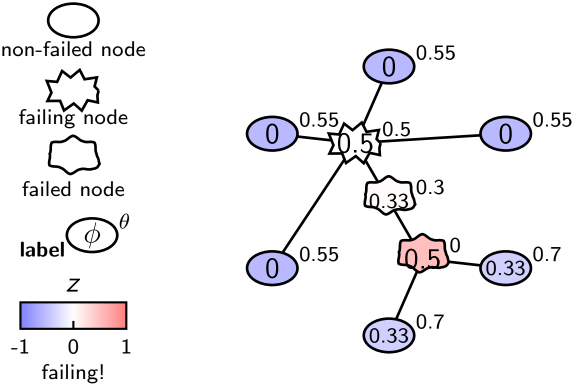

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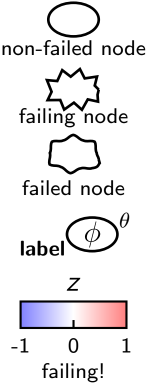
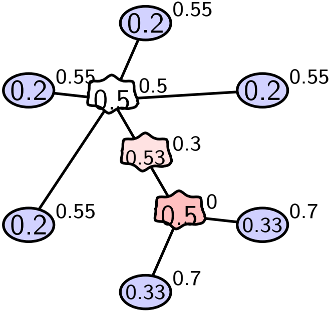
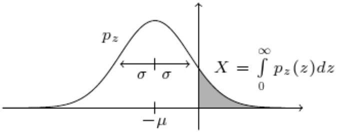


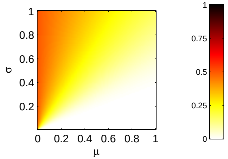


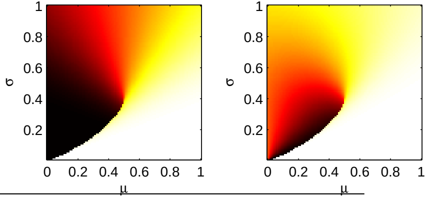
- low degree node causes more damage than in 'inward' variant
  - 'systemic risk' strongly depends on initial position, distributions

Systemic risk as a phase transition

- initial conditions normally distributed: $\theta \sim \mathcal{N}(-\mu, \sigma)$,
  - σ: measure of *initial heterogeneity* in θ across nodes
  - initial failure: $X(0) = \Phi_{\mu,\sigma}(0)$ (cumulative normal distribution)







- **First-order phase transition:** small variations in initial conditions lead to complete failure
- non-monotonous behavior: intermediate σ most dangerous
- **(right)** systemic risk resulting from *cascades* only

* J. Lorenz, S. Battiston, F.S. Eur Phys J B **71** (2009) 441-460

Conclusions

What did we learn about complex networks?

- distinct **topologies** and **growth** mechanisms
  - statistical **regularities** exist (degree distribution, ...)
- real networks: **weighted**, **directed**, **time** dependent
  - **backbones** of few nodes account for most of properties

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Challenges for research on complex networks

- feedback between **agent** and **link** dynamics
  - **policy implications**: how to regulate network structures
- emergence of **systemic properties** ($\rightarrow$ systemic risk)
  - relations between topology, susceptibility, redistribution mechanisms?
- understand **deviations** from universality
  - agents: strategic link formation/deletion $\rightarrow$ suboptimal solutions?