Entropy for complex systems

Stefan Thurner & Rudolf Hanel

www.complex-systems.meduniwien.ac.at www.santafe.edu

vienna nov 26 2010

C.E. Shannon, The Bell System Technical Journal **27**, 379-423, 623-656, 1948.

Appendix 2, Theorem 2

What are Complex Systems ?

- CS are made up from many elements
- These elements are in strong correlation/contact with each other
- CS strongly influence their own boundary conditions
- CS are often non-Markovian

Simple vs. Complex Systems ?

• Weakly interacting statistical systems: thermodynamics – given W large

• CS: long-range & strong interactions $\rightarrow$ change macroscopic qualitative properties as a function of the number of states (system size)

$\rightarrow$ extremely rich behavior of complex systems: assemblies of neurons, state forming insects, societies etc.

$\rightarrow$ large assemblies markedly different systemic- or macro properties than those composed of a few elements

Why talking of entropy of Complex Systems ?

- The central concept: understanding macroscopic system behavior on the basis of microscopic properties $\rightarrow$ *entropy*

- Entropy relates number of states to an extensive quantity, plays fundamental role in the thermodynamical description

- Hope: 'Thermodynamical' relations for CS, phase diagrams for CS, etc.

- Dream: some way to reduce number of parameters $\rightarrow$ handle CS

Entropy of interacting statistical systems

Two initially isolated systems: A and B with W_A and W_B states

Additive: entropy combined system $A+B$: $S(W_A W_B) = S(W_A) + S(W_B)$

Extensive: entropy combined system $A+B$: $S(W_{A+B}) = S(W_A) + S(W_B)$

Non-interacting: states in combined system $W_{A+B} = W_A W_B$

Non-interacting: $S_{\mathrm{BG}}[p] = -\sum_i -p_i \ln p_i$: additive and extensive

Interacting: $W_{A+B} \leq W_A W_B$ (non-ergodic)

In this case Boltzmann-Gibbs entropy is no longer extensive !!!

WANTED: extensive entropies

Why generalized entropies ?

To ensure extensivity of entropy in strongly interacting system

$\rightarrow$ find entropic form for particular system $\rightarrow$ generalized entropies

$$S_g[p] = \sum_{i=1}^{W} g(p_i) \qquad W \quad \dots \quad \text{number} \quad \text{of} \quad \text{states}$$

COSY Complex Systems Research Group
SANTA FE INSTITUTE

The Shannon-Khinchin axioms

- K1: S depends continuously on $p \to g$ is continuous
- K2: entropy maximal for equi-distribution $p_i = 1/W \to g$ is concave
- K3: $S(p_1, p_2, \cdots, p_W) = S(p_1, p_2, \cdots, p_W, 0) \to g(0) = 0$
- K4: $S(A + B) = S(A) + S(B|A)$

Theorem: If K1 to K4 hold, entropy is Boltzmann-Gibbs-Shannon entropy

$$S_{\mathrm{BG}}[p] = \sum_{i=1}^{W} g_{\mathrm{BG}}(p_i) \quad \text{with} \quad g_{\mathrm{BG}}(x) = -x \ln x$$

Shannon-Khinchin axiom 4 is non-sense for CS

K4 corresponds to Markovian processes or weak interactions
$\rightarrow$ violated for most interacting systems

- Assume axioms K1, K2, K3 and $S_g = \sum g(p)$

(K1-K3 is equivalent to: g is continuous, concave and $g(0) = 0$)

The Complex Systems axioms

- K1 holds
- K2 holds
- K3 holds
- $S_g = \sum_i^W g(p_i)$, $W \gg 1$

Theorem: If 4 axioms hold:

(1) all systems can be uniquely classified by 2 numbers, c and d.

(2) for all these systems there exists a unique entropy:

$$S_{c,d} = \frac{e}{1-c+cd}\left[\sum_{i=1}^{W} \Gamma\left(1+d\,,\,1-c\ln p_i\right) - \frac{c}{e}\right] \qquad e\cdots \mathrm{Euler\,const}$$

The argument

Assume no constraint on system $\rightarrow$ equi-distribution $p_i = \frac{1}{W}$

$$S_g = \sum_{i=1}^{W} g(p_i) = W g\left(\frac{1}{W}\right)$$

Study mathematical properties of g

- Scaling transformation $W \rightarrow \lambda W$: how does entropy change ?
- A second asymptotic property from specific scaling: $\lambda \rightarrow W^a$

Mathematical properties I: a scaling law

$$\lim_{W\to\infty} \frac{S_g(W\lambda)}{S_g(W)} = \lim_{W\to\infty} \lambda \frac{g(\frac{1}{\lambda W})}{g(\frac{1}{W})}$$

define scaling function

$$f(z) \equiv \lim_{x\to 0} \frac{g(zx)}{g(x)} \qquad (0 < z < 1)$$

Theorem 1: for systems satisfying K1, K2, K3

$\rightarrow f$ can only be a power $f(z) = z^c$, with $0 < c \leq 1$

Obviously: $\lim_{W\to\infty} \frac{S_g(\lambda W)}{S_g(W)} = \lambda^{1-c}$ Keep this in mind!

Theorem 1

Let g be a continuous, concave function on $[0,1]$ with $g(0) = 0$ and let $f(z) = \lim_{x \to 0^+} g(zx)/g(x)$ be continuous, then f is of the form $f(z) = z^c$ with $c \in (0,1]$.

Proof. Note that $f(ab) = \lim_{x \to 0} g(abx)/g(x) = \lim_{x \to 0}(g(abx)/g(bx))(g(bx)/g(x)) = f(a)f(b)$. All pathological solutions are excluded by the requirement that f is continuous. So $f(ab) = f(a)f(b)$ implies that $f(z) = z^c$ is the only possible solution of this equation. Further, since $g(0) = 0$, also $\lim_{x \to 0} g(0x)/g(x) = 0$, and it follows that $f(0) = 0$. This necessarily implies that $c > 0$. $f(z) = z^c$ also has to be concave since $g(zx)/g(x)$ is concave in z for arbitrarily small, fixed $x > 0$. Therefore $c \leq 1$. $\square$

Mathematical properties II: an asymptotic property

Substitute λ by $\lambda \to W^a$ $\to$ identify a second asymptotic property

Define

$$h_c(a) \equiv \lim_{W\to\infty} \frac{S(W^{1+a})}{S(W)} W^{a(c-1)} = \lim_{x\to 0} \frac{g(x^{1+a})}{x^{ac} g(x)} \qquad (x = \frac{1}{W})$$

$h_c(a)$ in principle depends on c and a, BUT

Theorem 2: Under K1-K3, $h_c(a)$ can only be

$$h_c(a) = (1+a)^d \qquad (d \text{ constant})$$

Remarkably, this is independent of c and $h_c(a)$ is an asymptotic property which is independent of first scaling property!

Note that if $c = 1$, concavity of g implies $d \geq 0$

Theorem 2

Let g be like in Theorem 1 and let $f(z) = z^c$ then h_c given in Eq. (8) is a constant of the form $h_c(a) = (1+a)^d$ for some constant d.

Proof. We determine $h_c(a)$ again by a similar trick as we have used for f.

$$\begin{aligned} h_c(a) &= \lim_{x\to 0} \frac{g(x^{a+1})}{x^{ac} g(x)} = \frac{g\left((x^b)^{\left(\frac{a+1}{b}-1\right)+1} \right)}{(x^b)^{\left(\frac{a+1}{b}-1\right)c} g(x^b)} \frac{g(x^b)}{x^{(b-1)c} g(x)} \\ &= h_c\left(\frac{a+1}{b}-1\right) h_c(b-1) \quad , \end{aligned}$$

for some constant b. By a simple transformation of variables, $a = bb' - 1$, one gets $h_c(bb'-1) = h_c(b-1)h_c(b'-1)$. Setting $H(x) = h_c(x-1)$ one again gets $H(bb') = H(b)H(b')$. So $H(x) = x^d$ for some constant d and consequently $h_c(a)$ is of the form $(1+a)^d$. □

Summary

Interacting systems $\to$ require K1-K3 and $S = \sum g(p_i)$

$\to f(z) = \lim_{x \to 0} \frac{g(zx)}{g(x)} = z^c$ $\quad 0 \leq c < 1$

$\to h_c(a) = \lim_{x \to 0} \frac{g(x^{1+a})}{x^{ac} g(x)} = (1+a)^d$ $\quad d$ real

Remarkable: all systems are characterized by a pair of 2 exponents: (c, d)

Examples

- Boltzmann-Gibbs: $g_{\mathrm{BG}}(x) = -x \ln(x)$

  $f(z) = z$, i.e. $c = 1$

  $h_1(a) = 1 + a$, i.e. $d = 1$

$\rightarrow S_{\mathrm{BG}}$ belongs to the universality class $(c, d) = (1, 1)$

- Tsallis: $g_q(x) = (x - x^q)/(1 - q)$

  $f(z) = z^q$, i.e. $c = q$

  $h_0(a) = 1$, i.e. $d = 0$

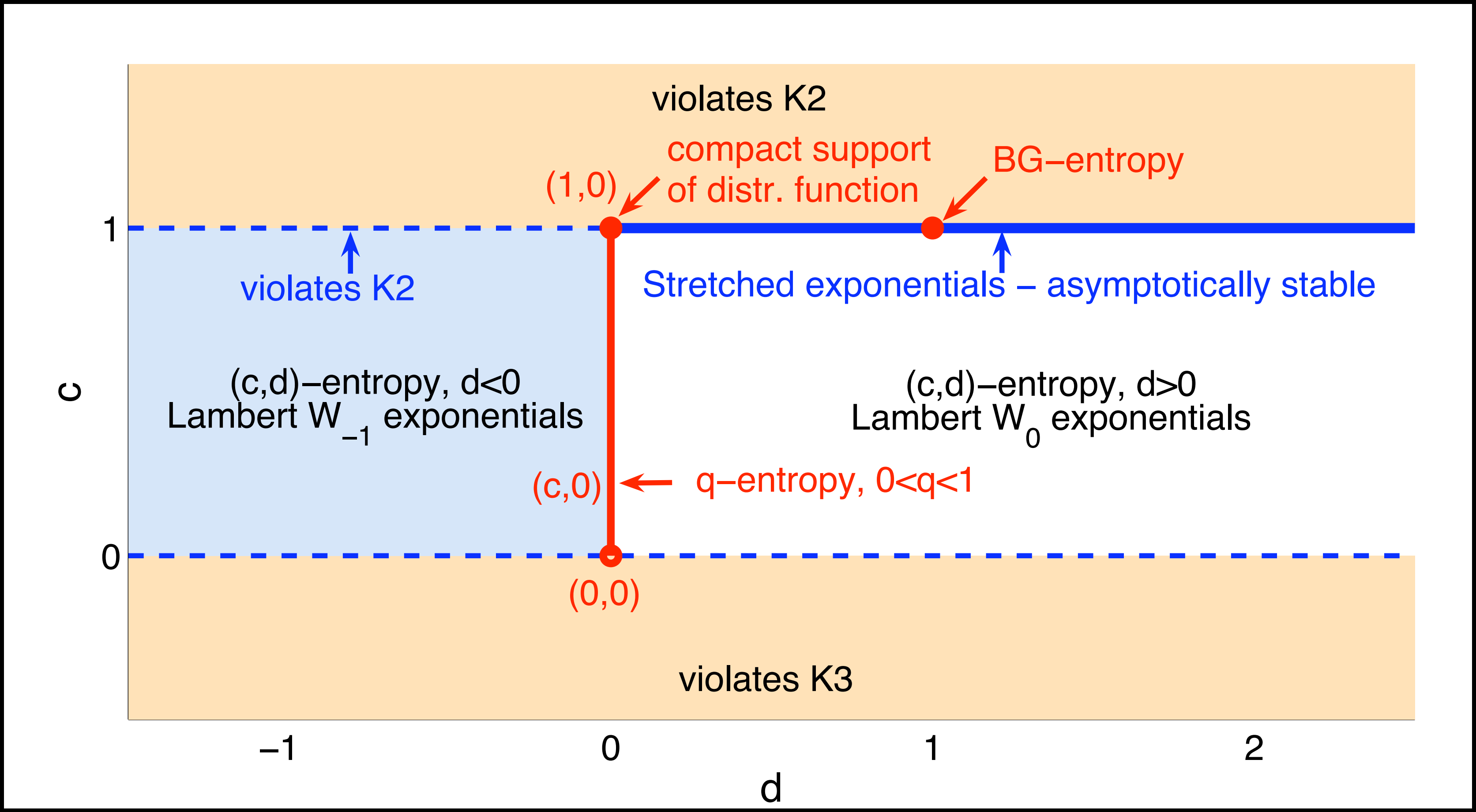
$\rightarrow S_q$ belongs to the universality class $(c, d) = (q, 0)$

- etc ...

Classification

entropy		c	d
$S_{BG} = \sum_i p_i \ln(1/p_i)$		1	1
• $S_{q<1} = \frac{1-\sum p_i^q}{q-1}$	$(q<1)$	$c = q < 1$	0
• $S_\kappa = \sum_i p_i (p_i^\kappa - p_i^{-\kappa})/(-2\kappa)$	$(0 < \kappa \leq 1)$	$c = 1 - \kappa$	0
• $S_{q>1} = \frac{1-\sum p_i^q}{q-1}$	$(q>1)$	1	0
• $S_b = \sum_i (1 - e^{-bp_i}) + e^{-}b - 1$	$(b>0)$	1	0
• $S_E = \sum_i p_i (1 - e^{\frac{p_i - 1}{p_i}})$		1	0
• $S_\eta = \sum_i \Gamma(\frac{\eta+1}{\eta}, -\ln p_i) - p_i \Gamma(\frac{\eta+1}{\eta})$	$(\eta > 0)$	1	$d = 1/\eta$
• $S_\gamma = \sum_i p_i \ln^{1/\gamma}(1/p_i)$		1	$d = 1/\gamma$
• $S_\beta = \sum_i p_i^\beta \ln(1/p_i)$		$c = \beta$	1

$S_{c,d} = \sum_i e r \Gamma(d+1, 1 - c \ln p_i) - cr$	c	d


violates K2
compact support
of distr. function
BG–entropy
(1,0)
1
violates K2
Stretched exponentials – asymptotically stable
c
(c,d)–entropy, d<0
Lambert W_−1 exponentials
(c,d)–entropy, d>0
Lambert W_0 exponentials
(c,0)
q–entropy, 0<q<1
0
(0,0)
violates K3
−1
0
1
2
d

The entropy

Question: which g fulfills $f(z) = z^c$ and $h_c(a) = (1+a)^d$?

Answer: $g_{c,d,r}(x) = re\,\Gamma\,(1+d\,,\,1-c\ln x) - rcx \rightarrow$

$$S_{c,d} = \sum_{i=1}^{W} re\,\Gamma\,(1+d\,,\,1-c\ln p_i) - rc \qquad\qquad r = \frac{1}{1-c+cd}$$

Γ ... incomplete Gamma function $\Gamma(a,b) = \int_b^\infty dt\, t^{a-1} \exp(-t)$

Proof: see Theorem 4

Examples

- $S_{1,1} = \sum_i g_{1,1}(p_i) = -\sum_i p_i \ln p_i + 1$ (BG entropy)
- $S_{c,0} = \sum_i g_{c,0}(p_i) = \frac{1-\sum_i p_i^c}{c-1} + 1$ (Tsallis entropy)
- $S_{1,d>0} = \sum_i g_{1,d}(p_i) = \frac{e}{d} \sum_i \Gamma\left(1+d\,,\,1-\ln p_i\right) - \frac{1}{d}$ (AP entropy)
- ...

Distribution functions of CS

Entropy $\rightarrow$ generalized logarithm $\rightarrow$ generalized exponential $\equiv$ distribution function:

$$p_{c,d}(x) = e^{-\frac{d}{1-c}\left[W_k\left(B(1+\frac{x}{r})^{\frac{1}{d}}\right) - W_k(B)\right]} \qquad B \equiv \frac{1-c}{cd}\exp\left(\frac{1-c}{cd}\right)$$

W_k... k'th branch of Lambert-W function: solution to $x = W(x)e^{W(x)}$

only branch $k = 0$ and $k = -1$ have real solutions

$d \geq 0$ $\rightarrow$ take branch $k = 0$

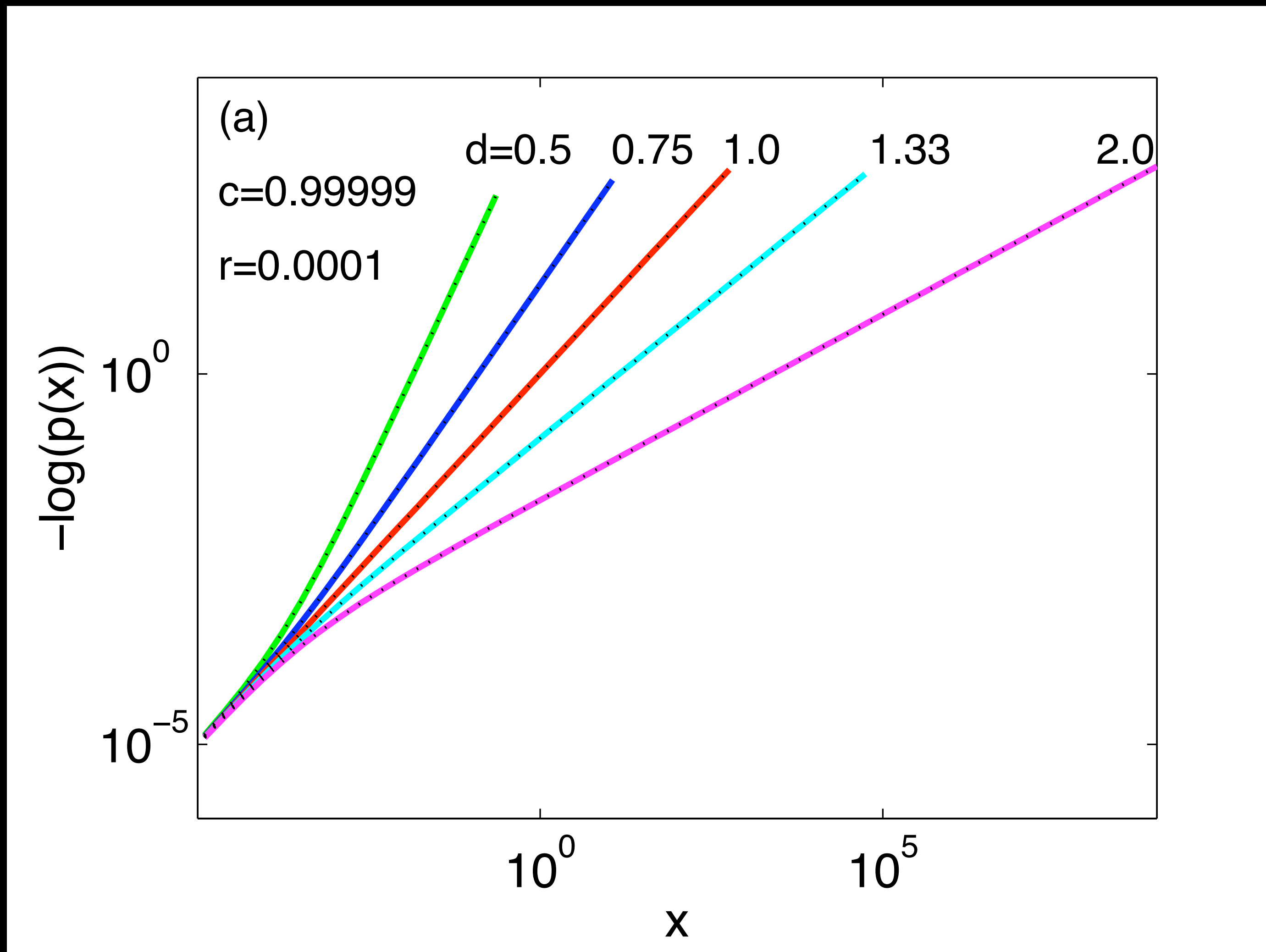
$d < 0$ $\rightarrow$ take branch $k = -1$

Distribution functions of CS

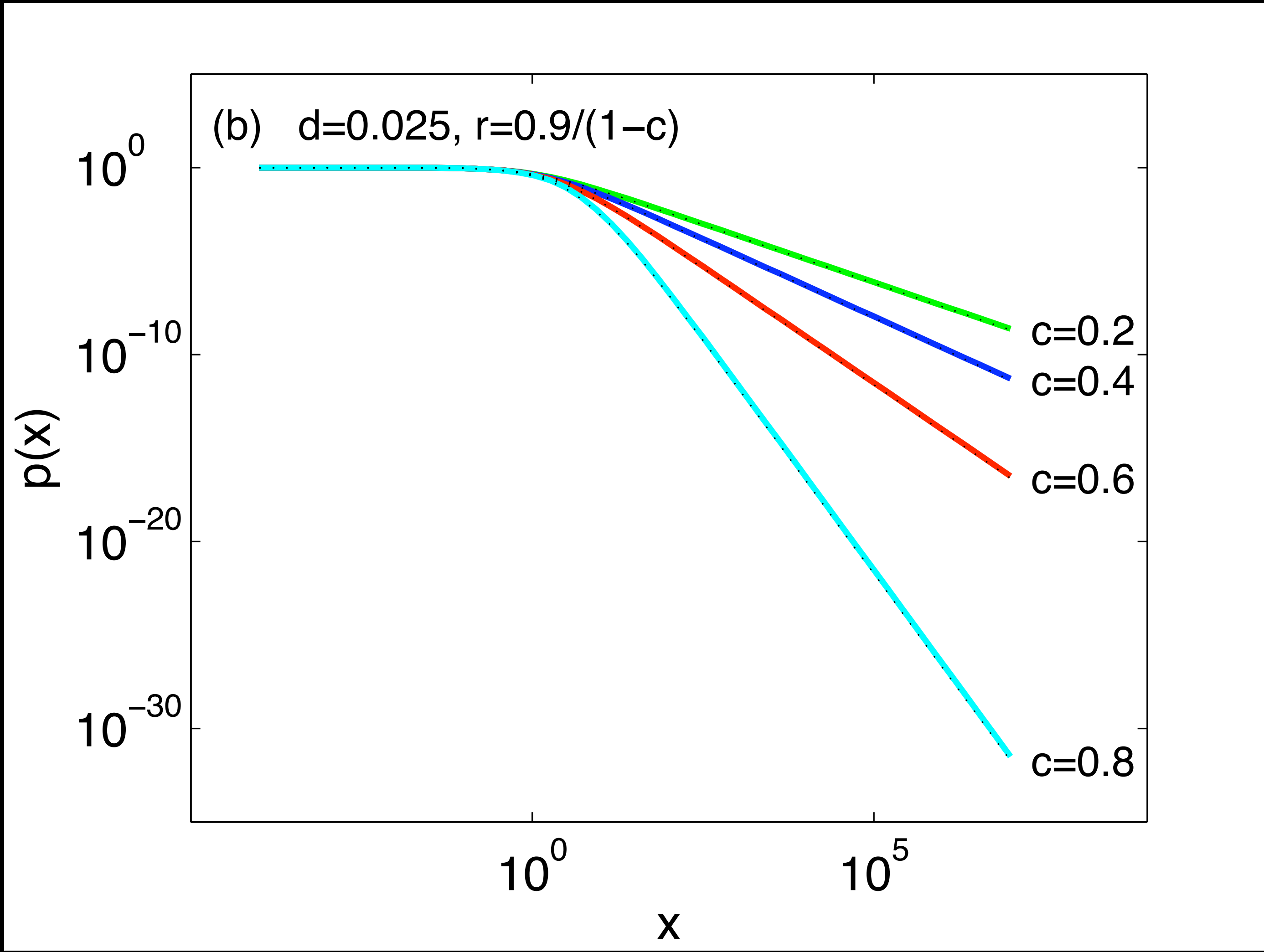
- $(c, d) = (1, 1) \rightarrow$ Boltzmann distribution
- $(c, d) = (q, 0) \rightarrow$ power-laws (q-exponentials)
- $(c, d) = (1, d)$, for $d > 0 \rightarrow$ stretched exponentials
- (c, d) all others $\rightarrow$ Lambert-W exponentials

NO OTHER POSSIBILITIES

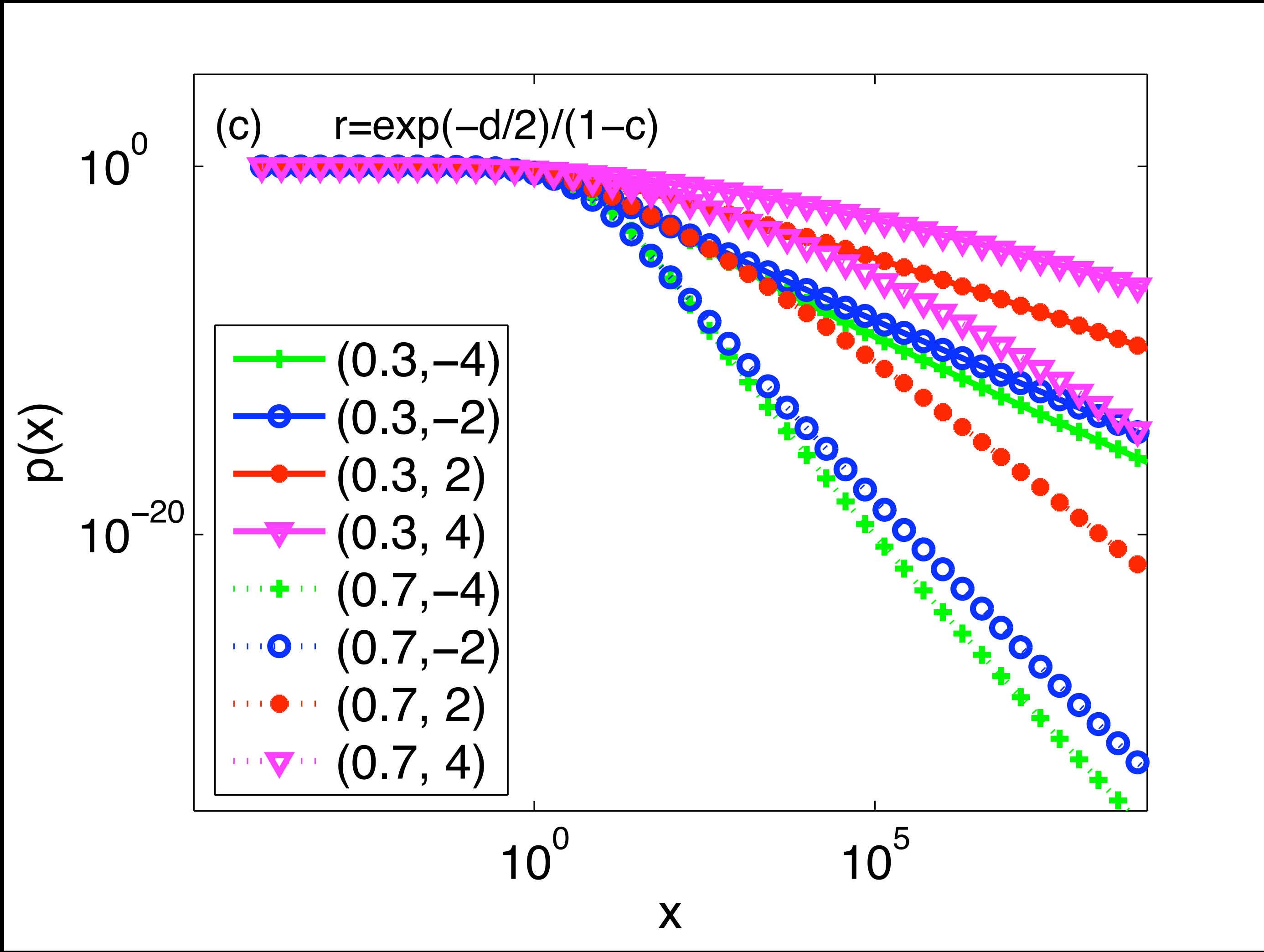
Streched Exponential: $c = 1$, $d > 0$

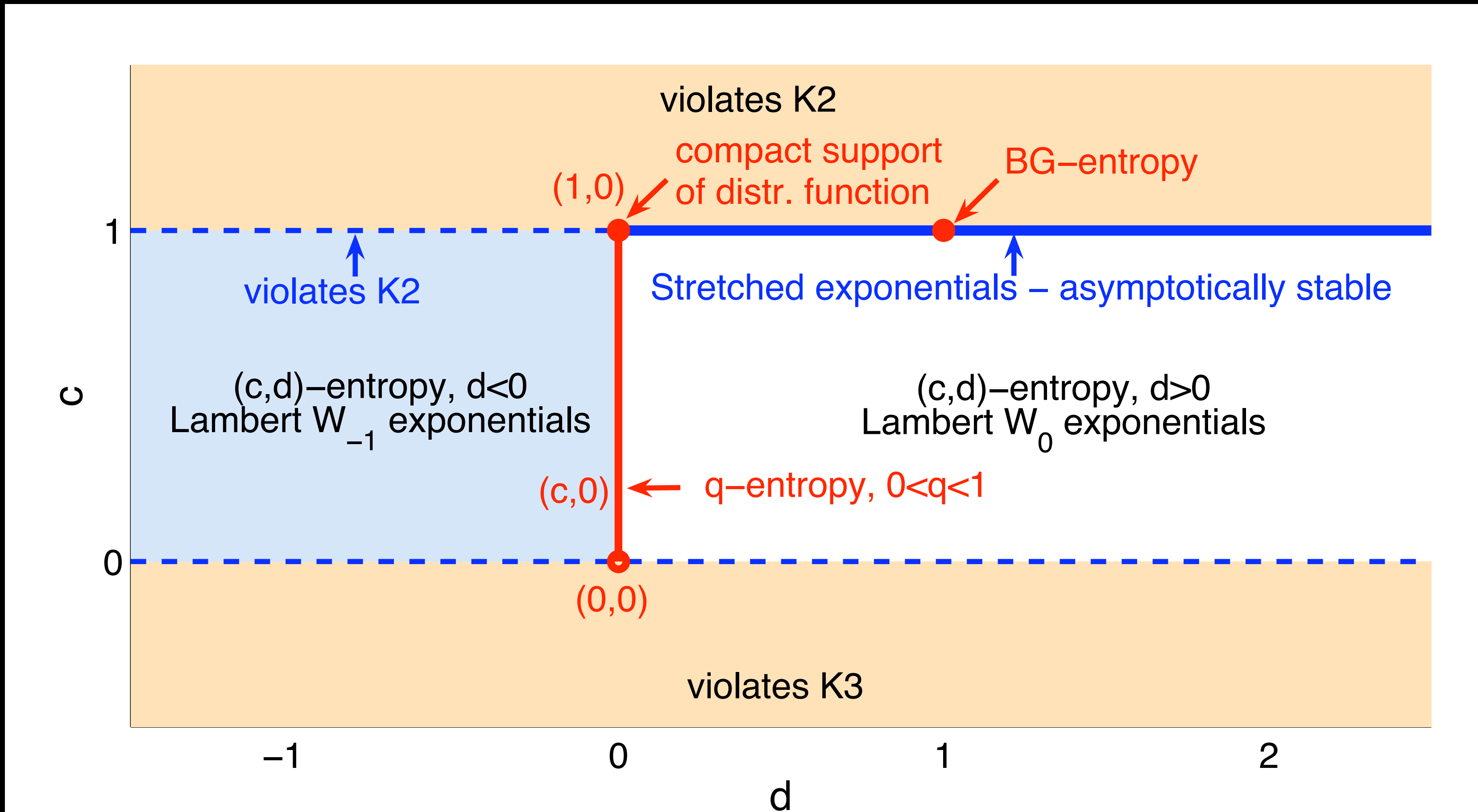


q-exponentials: $0 < c \leq 1$, $d = 0$



Lambert-W




violates K2
compact support
of distr. function
BG–entropy
(1,0)
1
violates K2
Stretched exponentials – asymptotically stable
c
(c,d)–entropy, d<0
Lambert W$_{-1}$ exponentials
(c,d)–entropy, d>0
Lambert W$_{0}$ exponentials
(c,0)
q–entropy, 0<q<1
0
(0,0)
violates K3
–1
0
1
2
d

Example: a physical system

equation of motion for particle i in system of N overdamped particles

$$\mu \vec{v}_i = \sum_{j \neq i} \vec{J}(\vec{r}_i - r_j) + \vec{F}(\vec{r}_i) + \eta(\vec{r}_i, t)$$

v_i ... velocity of i th particle μ ... viscosity of medium F ... external force

$\vec{J}(\vec{r}) = G\left(\frac{|\vec{r}|}{\lambda}\right) \hat{r}$ ... repulsive particle-particle interaction

η ... uncorrelated thermal noise $\langle \eta \rangle = 0$ and $\langle \eta^2 \rangle = \frac{kT}{\mu}$

λ ... characteristic length of short range pairwise interaction

Shown with FP approach and simulation (arXiv: 1008.1421v1)

- low temperature: Tsallis system $(c, d) = (q, 0)$
- high temperature limit $\rightarrow$ BG system $(c, d) = (1, 1)$

A note on Rényi entropy

It is it not sooo relevant for CS. Why?

- Relax Khinchin axiom 4:

$$S(A+B) = S(A) + S(B|A) \rightarrow S(A+B) = S(A) + S(B) \rightarrow \text{Rényi entropy}$$

- $S_R = \frac{1}{\alpha-1} \ln \sum_i p_i^\alpha$ violates our $S = \sum_i g(p_i)$

But: our above argument also holds for Rényi-type entropies !!!

$$S = G\left(\sum_{i=1}^{W} g(p_i)\right)$$

$$\lim_{W\to\infty} \frac{S(\lambda W)}{S(W)} = \lim_{R\to\infty} \frac{G\left(\frac{f_g(z)}{z} G^{-1}(R)\right)}{R} = [\text{for } G \equiv \ln] = 1$$

Bonus track:
A note on finite systems

Told you: $r = \frac{1}{1-c+cd}$. This is not the most general case !

Can pick r freely – as long as

$$\begin{array}{ll} d > 0: & r < \frac{1}{1-c} \\ d = 0: & r = \frac{1}{1-c} \\ d < 0: & r > \frac{1}{1-c} \end{array}$$

then the corresponding generalized logarithms $\Lambda(p(-x)) = x$ have the usual properties: $\Lambda(1) = 0$ and $\Lambda'(1) = 1$

- every choice of r gives a representative of the equivalence class (c, d)
- r encodes finite-size characteristics of distribution

Conclusions

- Interpret CS as those where Khinchin axioms 1-3 hold and $S = \sum g$
- Showed: macroscopic statistical systems can be uniquely classified in terms of their asymptotic ($W \gg 1$) properties
- Systems classified by two exponents (c, d) – analogy to critical exponents
- (c, d) define equivalence relations on entropic forms
- Single entropy covers all systems: $S_{c,d} = re \sum_i \Gamma\,(1 + d\,,\, 1 - c \ln p_i) - rc$
- All known entropies of admissible systems are special cases
- Distribution functions of *all* systems belong to class of Lambert-W exponentials. There are no other options
- Remarkable: Tsallis case sandwiched between the 2 Lambert solutions