Two-loop solution of the O(N) model from 2PI effective potential

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6^{th} Vienna Central European Seminar
27-29 November, 2009

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Outline

O(N) model as an effective theory

2PI formalism and its approximations

Renormalisation at 2-loop level

Algorithm for solving the finite equations at zero temperature

Summary

O(N) model as an effective theory

- One of the most popular effective theory of QCD is the $SU(N) \times SU(N)$ linear sigma model
  $\longrightarrow$ Dynamical variable is a matrix $\in U(N)$ Lie algebra
- Considering the $N = 2$ case (only u and d quarks are present) one can classify the fields as representations of the two $SU(2)$ groups $\Longrightarrow SU(2) \times SU(2) \simeq O(4)$
  $\longrightarrow \phi \equiv (\sigma, \vec{\pi}), \;\; \chi \equiv (\eta^{'}, \vec{a_0})$

$$L = \frac{1}{2}(\partial_\mu \phi^T \partial^\mu \phi + m_\phi^2 \phi^T \phi) - \frac{\lambda}{24}(\phi^T \phi)^2 + \frac{1}{2}(\partial_\mu \chi^T \partial^\mu \chi + m_\chi^2 \chi^T \chi) - \frac{\lambda}{24}(\chi^T \chi)^2 + L_i(\phi, \chi)$$

O(N) model as an effective theory

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- η' and a_0 masses are large $\longrightarrow$ these fields can be neglected
  $\Longrightarrow$ we are left with a theory with one $O(4)$ vector
- Symmetry can be extended to O(N)

2PI formalism

- In 2PI formalism[1] one has a generalised effective potential:

$$V_{2PI}[v,G] = V_{cl}[v] - \frac{i}{2}\int_p \left[\mathrm{Tr}\ln G^{-1}(p) + \mathrm{Tr}\left(D^{-1}(p)G(p)\right)\right] +$$
$$+ \text{ every 2PI skeleton diagram in the theory}$$

- Stationary conditions

$$\frac{\delta V}{\delta v} = 0, \quad \frac{\delta V}{\delta G} = 0$$

  give the equation(s) of state and the self-consistent Dyson-equation of the propagator(s)

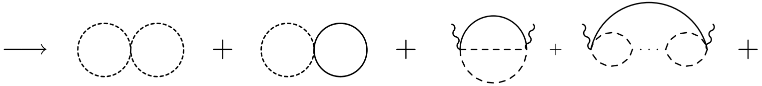
- The functional contains infinitely many skeletons
  $\longrightarrow$ restriction is needed to finite number of diagrams
  $\Longrightarrow$ partial resummation in the perturbation series

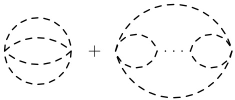
- Two main approximation schemes for the O(N) model:
  $\longrightarrow$ large N expansion and loop truncation

[1] J.M. Cornwall, R. Jackiw, E. Tomboulis, Phys. Rev. D10, 2428 (1974)

2PI formalism: large N approximation of the O(N) model

- Large N approximation of the 2PI potential at next-to-leading order[1] ($\mathcal{O}(N^0)$):





$\longrightarrow$ solid: sigma, dashed: pion
wiggly: mean field

[1] D.Dominici & U. Marini Bettolo Marconi, Phys. Lett. **B319**, 171 (1996)

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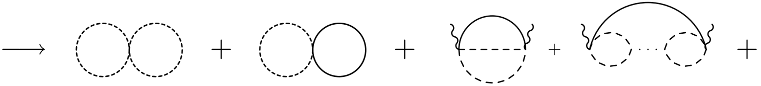

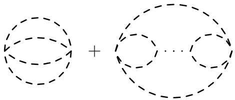

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- In the expansion the propagators and the mean field are treated as power series in $1/N$
  $\longrightarrow$ one loses self-consistency $\Rightarrow$ not really 2PI
  $\longrightarrow$ DS eq.'s also give the NLO equations: close the hierarchy with the 3-point function set to its tree-level value

[1] D.Dominici & U. Marini Bettolo Marconi, Phys. Lett. **B319**, 171 (1996)

- Some papers[1, 2] appeared which were raising doubts on renormalisability of the large N approximation for arbitrary values of the mean field

  ⟶ these statements turned out to be wrong

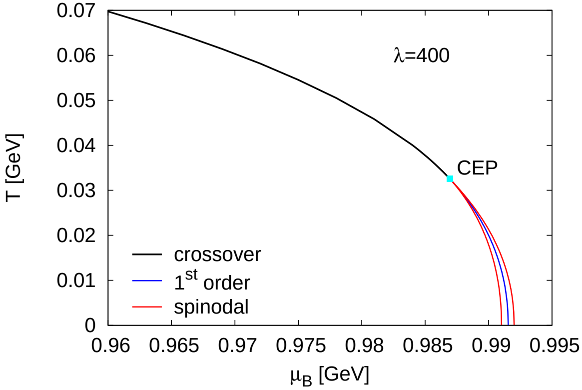
  ⟶ counterterms were calculated explicitly at NLO[3] (also in auxiliary field formalism)

[1] J.O. Andersen, D. Boer, H.J. Warringa, Phys. Rev. D70, 116007 (2004)
[2] J.O. Andersen, T. Brauner, Phys. Rev. D78, 014030 (2008)
[3] G. Fejős, A. Patkós, Zs. Szép, Phys. Rev. D 80, 025015 (2009)

2PI formalism: large N approximation of the O(N) model

- $SU(2) \times SU(2) \simeq O(4)$ model can be extended with quarks
  $\rightarrow$ the generalised model was investigated[1] at LO in large N

a., $iG_\pi^{-1}(p) = p^2 - m^2 - \Sigma(0)$



[1] G. Markó, Zs. Szép, work in progress

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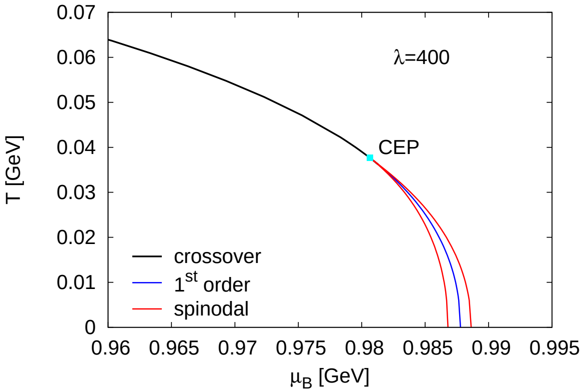
$$b., \quad iG_\pi^{-1}(p) = p^2 - m^2 - \Sigma(m_p^2)$$

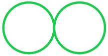
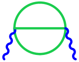


[1] G. Markó, Zs. Szép, work in progress

2PI formalism: 2 loop truncation of the O(N) model

- 2-loop truncation of the 2PI potential of the O(N) model:[1]

$$V_{2PI}[v,G]=\frac{1}{2}m^2v^2+\frac{\lambda}{24N}v^4-\frac{i}{2}\int_p\left[\mathrm{Tr}\ln G^{-1}(p)+\mathrm{Tr}\left(D^{-1}(p)G(p)\right)\right]$$

$+$  $+$  $+ V_{ct}$

- The complete two-loop truncation was motivated by the analysis of the double-bubble (Hartree) truncation
  $\Longleftarrow$ incorrect solution (1^{st} order phase transition) of the EoS at finite temperature for the $O(4)$ model

- First task: determine the counterterm functional

[1] A. Arrizabalaga & U. Reinosa: Nucl. Phys. A785 234 (2007)

Renormalisation at 2-loop level

- Perturbation series of the counterterms are also resummed $\longrightarrow$ due to the 2PI truncation counterterms related to different definitions of the 2- and 4-point functions do not coincide $\Rightarrow$ most general O(N) symmetric functional is needed[1]
- The self-interacting term must be expressed as a sum of O(N) invariant tensors:

$$\frac{\lambda}{24N}\Phi_a\Phi_a\Phi_b\Phi_b = \frac{\lambda}{24N}\frac{\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}}{3}\Phi_a\Phi_b\Phi_c\Phi_d$$

- The related piece of the counterterm functional:

$$V_{ct}^{(4)} = \frac{1}{24N}\frac{\delta\lambda_4^A\delta_{ab}\delta_{cd}+\delta\lambda_4^B[\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}]}{3}\Phi_a\Phi_b\Phi_c\Phi_d$$

- Classical- and quantum-parts should also contain different counterterms: 6 counterterms for λ, 2 for m^2

[1] A. Patkós & Zs. Szép: Nucl. Phys. A 811 329 (2008)

Renormalisation at 2-loop level

- The complete counterterm functional:

$$V_{ct} = \frac{1}{2}\delta m_0^2 v^2 + \frac{\delta\lambda_4^A + 2\lambda_4^B}{72N} v^4 +$$

$$\frac{1}{2}\left(\delta m_2^2 + \frac{\delta\lambda_2^A}{6N} v^2\right) \int_p \Big[G_\sigma(p) + (N-1) G_\pi(p)\Big] + \frac{\delta\lambda_2^B v^2}{6N} \int_p G_\sigma(p) +$$

$$\frac{\delta\lambda_0^A}{24N}\Big[\int_p [(N-1) G_\pi(p) + G_\sigma(p)]\Big]^2 +$$

$$\frac{\delta\lambda_0^B}{12N}\Big[(N-1)\Big(\int_p G_\pi\Big)^2 + \Big(\int_p G_\sigma\Big)^2\Big] - \frac{\delta Z}{2}\int_p p^2 \Big[(N-1) G_\pi(p) + G_\sigma(p)\Big]$$

Renormalisation at 2-loop level

- Self-consistent equations for the self energies:

$$\Sigma_\pi = m^2+\delta m_2^2+(\lambda+\delta\lambda_2^A)\frac{v^2}{6N}+\frac{1}{6N}[(N+1)\lambda+(N-1)\delta\lambda_0^A+2\delta\lambda_0^B]\,T_\pi$$
$$+\frac{1}{6N}(\lambda+\delta\lambda_0^A)\,T_\sigma+\frac{\lambda^2v^2}{9N^2}I_{\sigma\pi}(p)$$
$$\Sigma_\sigma = m^2+\delta m_2^2+(3\lambda+\delta\lambda_2^A+2\delta\lambda_2^B)\frac{v^2}{6N}+\frac{1}{6N}(3\lambda+\delta\lambda_0^A+2\delta\lambda_0^B)\,T_\sigma$$
$$+\frac{N-1}{6N}(\lambda+\delta\lambda_0^A)\,T_\pi+\frac{\lambda^2v^2}{18N^2}[9I_{\sigma\sigma}(p)+(N-1)I_{\pi\pi}(p)]$$

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$$+\frac{N-1}{6N}(\lambda+\delta\lambda_0^A)T_\pi+\frac{\lambda^2v^2}{18N^2}[9I_{\sigma\sigma}(p)+(N-1)I_{\pi\pi}(p)]$$

- Split the tadpole integrals into two parts: $T = T_F + T_{div}$
  $\longrightarrow$ Use the finite equations in the T integrals
- Find divergencies proportional to v^0, v^2, $T_{\pi,F}$ and $T_{\sigma,F}$
  $\Rightarrow$ 5 indep. equations determine $\delta\lambda_0^A, \delta\lambda_0^B, \delta\lambda_2^A, \delta\lambda_2^B$ and δm_2^2
- EoS determines δm_0^2 and only the combination of $\delta\lambda_4^A+2\delta\lambda_4^B$

Solving the equations

- For simplicity let us consider the $N = 1$ case
- With explicit counterterms one can solve the renormalised equations for the self energy and the mean field

$$0 = m^2 + \delta m_0^2 + \frac{1}{6}(\lambda + \delta\lambda_4)v^2 + \frac{1}{2}(\lambda + \delta\lambda_2)T(G) + \frac{1}{6}\lambda^2 S(0, G)$$

$$\Sigma(p) = \delta m_2^2 + \frac{1}{2}\delta\lambda_2 v^2 + \frac{1}{2}(\lambda + \delta\lambda_0)T(G) + \frac{1}{2}\lambda^2 v^2 I(p, G)$$

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- For finite temperature calculations one can work in imaginary time formalism:

$$k = (k_0, \mathbf{k}) \longrightarrow (i\omega_n, \mathbf{k}), \qquad \int_k \longrightarrow iT \sum_n \int_{\mathbf{k}}$$

$$G(k_0, \mathbf{k}) \longrightarrow \Delta(i\omega_n, \mathbf{k})$$

- Bosonic Matsubara frequencies: $\omega_n = 2\pi n/\beta$

Solving the equations

- In order to perform Matsubara sums, one can use the spectral representation of the propagator:

$$\Delta(i\omega_n, \mathbf{k}) = \int_{-\infty}^{+\infty} dk_0 \frac{\rho(k_0, \mathbf{k})}{k_0 - i\omega_n}$$

- Connection between the spectral function and the propagator:

$$\rho(k_0, \mathbf{k}) = -2\Im(-iG(k_0, \mathbf{k}))\epsilon(k_0)$$

- The sums in the quantum corrections:

$$T \rightarrow \frac{1}{\beta}\sum_n \frac{1}{k_0 - i\omega_n} = \frac{1}{e^{\beta k_0} - 1} \equiv f(k_0)$$

$$I(i\omega_e, \mathbf{p}) \rightarrow \frac{1}{\beta}\sum_m \frac{1}{(k_0 - i\omega_n)(q_0 - i(\omega_e - \omega_n))} =$$

$$= -\frac{f(-k_0) - f(q_0)}{k_0 + q_0 - i\omega_e}$$

Solving the equations

- After performing the sums:
  $\longrightarrow$ 3-momentum integrals can be calculated $\Rightarrow$ they are finite
  $\longrightarrow$ frequency integrals diverge

$$T \rightarrow \int_{-\infty}^{+\infty} dk_0 \int d^3k\ \rho(k_0, \mathbf{k})$$

$$I \rightarrow \int_{-\infty}^{+\infty} dk_0 \int_{-\infty}^{+\infty} dq_0 \int d^3k \int d^3q\ \rho(q_0, \mathbf{q})\rho(k_0, \mathbf{k})$$

  $\longrightarrow$ Regularization: cut-off in the frequency space: $|k_0| < \Lambda$
- Counterterms must also be calculated in this regularization
  $\longrightarrow$ they do not coincide with the ones calculated in other regularizations

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$$T \to \int_{-\infty}^{+\infty} dk_0 \int d^3k \; \rho(k_0, \mathbf{k})$$

$$I \to \int_{-\infty}^{+\infty} dk_0 \int_{-\infty}^{+\infty} dq_0 \int d^3k \int d^3q \; \rho(q_0, \mathbf{q})\rho(k_0, \mathbf{k})$$

  $\longrightarrow$ Regularization: cut-off in the frequency space: $|k_0| < \Lambda$
- Counterterms must also be calculated in this regularization
  $\longrightarrow$ they do not coincide with the ones calculated in other regularizations
- The spectral function is connected to the self-energy through the propagator: $\rho(k_0, \mathbf{k}) = -2\Im(-iG(k_0, \mathbf{k}))\epsilon(k_0)$
  $\longrightarrow$ using this relation one obtains a self-consistent equation for the Σ self-energy: solve it iteratively

Solving the equations

- Let us consider the $T = 0$ solution of the spectral function:
  $\longrightarrow$ assuming, that a stable quasi-particle appears at
  $p^2 = M_P^2$, the function splits into two pieces:

$$\rho(p_0, \mathbf{p}) = \rho_0(p_0, \mathbf{p}) + \delta\rho(p_0, \mathbf{p})$$

- ρ_0 comes from the pole of the propagator:
  $\rightarrow \rho_0(p_0, \mathbf{p}) = 2\pi Z \delta(p_0^2 - \mathbf{p}^2 - M_p^2)\varepsilon(p_0)$ with
  $$Z = (1 - \Sigma'(p^2 = M_p^2))^{-1}$$

- The $\delta\rho$ is the continuum part of the spectral function
  $\longrightarrow$ it begins at the two-particle threshold: $p_0^2 - p^2 > 4M_p^2$

Solving the equations

- Using the $\rho = \rho_0 + \delta\rho$ splitting of the spectral function:

$$\Sigma(p_0,\mathbf{p}) = \Sigma_0(p_0,\mathbf{p}) + \frac{1}{2}(\lambda+\delta\lambda_0)\delta T + \frac{1}{2}\lambda^2 v^2 \delta I(p_0,\mathbf{p})$$

- The iteratively devolping terms are:

$$\delta T = \frac{1}{4\pi^3}\int_0^{\Lambda} dq_0 \int_0^{\infty} dk q^2 \delta\rho(q_0,q),$$

$$\Im\delta I(p_0,p) = -\frac{1}{8\pi^2 p}\int_0^{p_0-M_p} dq_0 \int_0^{\infty} q\delta\rho(q_0,q) + \dots$$

$$\Re\delta I(p_0,p) = -\frac{1}{16\pi^3 p}\int_0^{\Lambda} dq_0\ \mathcal{P}\int_0^{\infty} dq_1 \Big[\frac{q_1}{(p_0+q_0)^2-q_1^2-M_p^2} + \\ + \frac{q_1}{(p_0-q_0)^2-q_1^2-M_p^2}\Big]\int_{|p-q_1|}^{p+q_1} dq_2 q_2 \delta\rho(q_0,q_2) + \dots$$

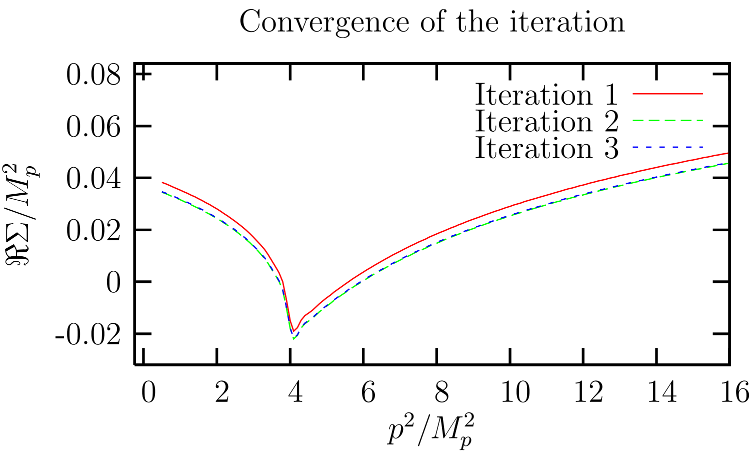
Solving the equations

- Starting the iteration with $\Sigma = \Sigma_0$
  $\longrightarrow \rho(p_0, \mathbf{p}) = \rho_0(p_0, \mathbf{p})$ with $Z = 1$
- The EoS and the propagator equation must be solved simultaneously with the constraint

$$M_p^2 = m^2 + \frac{\lambda}{2} v^2 + \Re\Sigma(p^2 = M_p^2)$$

- M_p^2 is fixed, m^2 is not arbitrary
  $\longrightarrow$ one has to treat m^2 as a variable in addition to v and Σ
- The renormalised mass parameter develops during the iteration procedure $\longrightarrow$ so do the mass counterterms

Numerical results



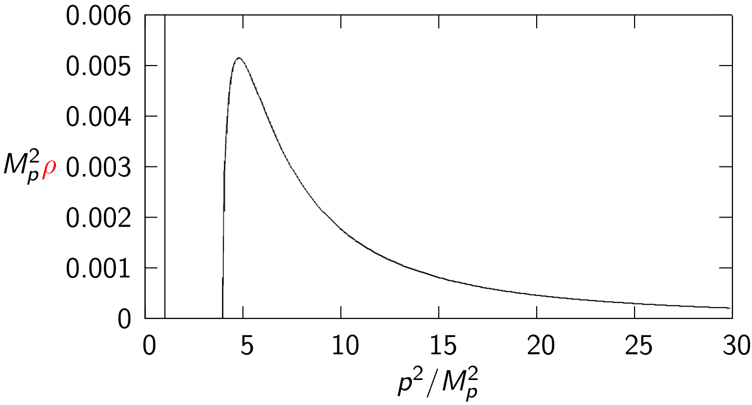
- The plot is made with the parameters: $\Lambda = 20.0 M_p$, $M_0 = 2.1 M_p$, $\lambda = 3.0$

$\longrightarrow$ the convergence is very fast

$\longrightarrow$ the non-perturbative effect is small but not negligible

Numerical results

Spectral function with $M_0 = 2.1M_p$, $\Lambda = 20.0M_p$



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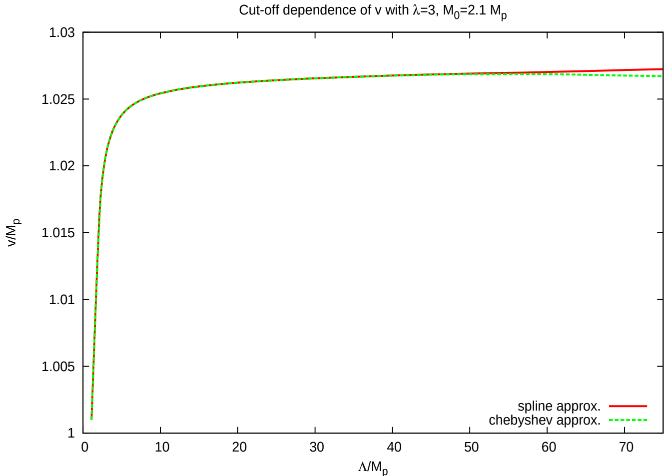

- Cut-off independence of v is true up to 0.1%
  $\longrightarrow$ slight logarithmic Λ dependence still remains
  $\longrightarrow$ numerical instability shows up for $\Lambda \gtrsim 60 M_p$

Summary

- Analysis of the complete two-loop truncation of the 2PI effective action in the $O(N)$ model
- Renormalisation was made with explicit counterterm construction
- Renormalised equations were solved numerically using iterative method $\longrightarrow$ fast convergence
- Cut-off dependence is neglegible for physical applications
- Near future:
  $\longrightarrow$ increase numerical stability for large cut-offs
  $\longrightarrow$ reduction of the remaining logarithmic divergence
  $\longrightarrow$ finite temperature calculations, comparing results with covariant spectral function method
- Describe phase transition in SU(3)$\times$SU(3) meson model
- Investigations in the quark-meson model with confining gauge background (Polyakov loop)