Symbolic Derivation of Dyson-Schwinger Equations using *Mathematica*

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SIC!QFT

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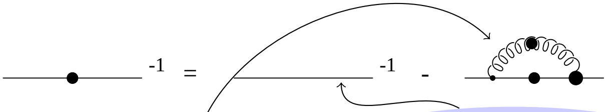

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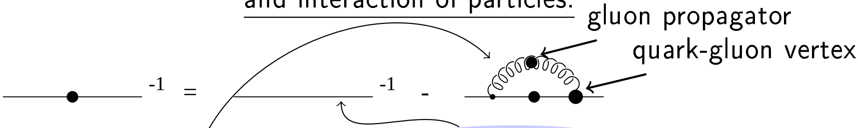

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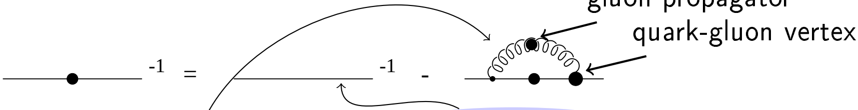

- Propagation of a quark: nothing happens or it emits/absorbs a gluon "in all possible ways" $\longrightarrow$ dressed propagators and vertices
- Unknowns: Quark-gluon vertex, gluon propagator
- Spontaneous chiral symmetry breaking
  $\Rightarrow$ dynamically generated momentum dependent quark masses

Dyson-Schwinger Equations (DSEs) for Investigating QCD

Facts about DSEs

- F. J. Dyson (1949) and J. S. Schwinger (1951)
- Equations of motion of Green functions
- Infinitely large tower of equations (DSE for n-point function contains n+1- and n+2-point functions)

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  → non-perturbative regime acessible
- Continuum
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- Truncations (not for all tasks)
- Gauge-dependent
  → Exploit advantages of different gauges

How to Derive a DSE? (I)

ϕ: generic field (quarks, gluons, ghosts, scalars, ...).
Put all indices (Lorentz, color), coordinates and field type in one index.
Integration and summation understood.

Start with the path integral. Integral of a total derivative vanishes (given appropriate boundary conditions):

$$Z[J] = \int D[\phi] e^{-S[\phi]+J\phi} \longrightarrow \quad \frac{\delta Z}{\delta \phi_i} = \left(-\frac{\delta S}{\delta \phi'_i}\bigg|_{\phi'_i = \delta/\delta J_i} + J_i \right) Z[J] = 0$$

$$-\frac{\delta S}{\delta \phi'_i}\bigg|_{\phi'_i = \delta W/\delta J_i + \delta/\delta J_i} + J_i = 0$$ connected n-point functions

$$-\frac{\delta S}{\delta \phi'_i}\bigg|_{\phi'_i = \phi_i + D_{ij}\delta/\delta \phi_j} + \frac{\delta \Gamma}{\delta \phi_i} = 0$$ **1PI n-point functions**

How to Derive a DSE? (II)

Continue by applying derivatives with respect to the fields:

$$\frac{\delta^n \Gamma}{\delta \phi_{i_1 \dots i_n}} = \frac{\delta S}{\delta^n \phi'_{i_1 \dots i_n}}\Bigg|_{\phi'_i = \phi_i + D_{ij}\delta/\delta\phi_j}$$

$\phi = 0$ only at the end
$\Rightarrow$ field-dependent intermediate Green functions.

Number of terms grows with n and number of interactions!

1 field, 3- and 4-point interactions:

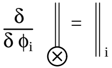
	# diagrams
2-point	5
3-point	12
4-point	53
5-point	359
...	...

Automatization possible?

Diagrammatic Rules

What does applying a derivative mean

for a field? $\dfrac{\delta \phi_j}{\delta \phi_i} = \delta_{ij}$



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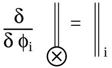
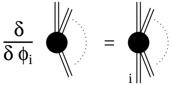


for a vertex? $\frac{\delta}{\delta\phi_i}\Gamma_{j_1\dots j_n} = \frac{-\delta^{n+1}\Gamma}{\delta\phi_i\delta\phi_{j_1}\dots\delta\phi_{j_n}} = \Gamma_{ij_1\dots j_n}$



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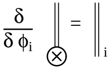
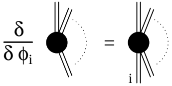
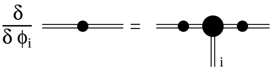


for a vertex? $\frac{\delta}{\delta \phi_i} \Gamma_{j_1 \ldots j_n} = \frac{-\delta^{n+1}\Gamma}{\delta \phi_i \delta \phi_{j_1} \ldots \delta \phi_{j_n}} = \Gamma_{i j_1 \ldots j_n}$



for a propagator? $\frac{\delta}{\delta \phi_i} D_{jk} = \frac{\delta}{\delta \phi_i} \left(\frac{\delta^2 \Gamma}{\delta \phi_j \delta \phi_k} \right)^{-1} = D_{jj'} \Gamma_{j' i k'} D_{k' k}$



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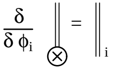

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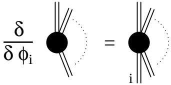

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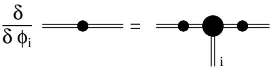
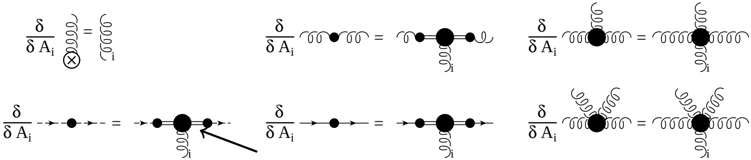


Ex.: Derivative with respect to gluon in Landau gauge QCD



generic field!

Example: Landau Gauge Yang-Mills theory

$$\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

Gluonic self-interaction!

Gauge fixed to Landau gauge.

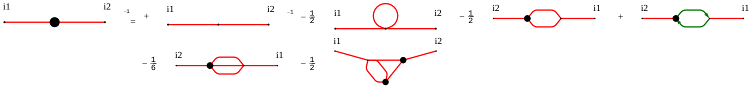
- 2 propagators: gluons A and ghosts c
- 3 interactions: AAA, Acc; $AAAA$

We start from

$$\frac{\delta^n \Gamma}{\delta \phi_{i_1 \ldots i_n}} = \left. \frac{\delta S}{\delta^n \phi'_{i_1 \ldots i_n}} \right|_{\phi'_i = \phi_i + D_{ij} \delta / \delta \phi_j}$$

Landau Gauge: Propagators

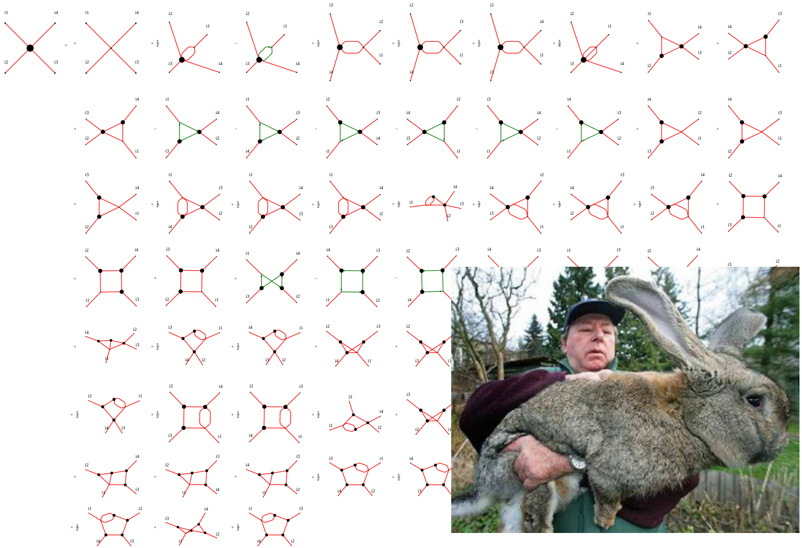
Gluon propagator:



Ghost propagator:

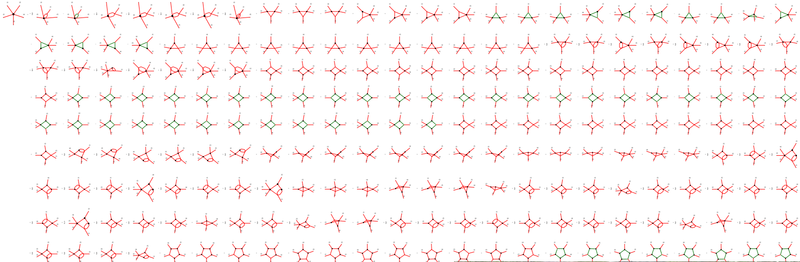
Landau Gauge: Four-Gluon Vertex

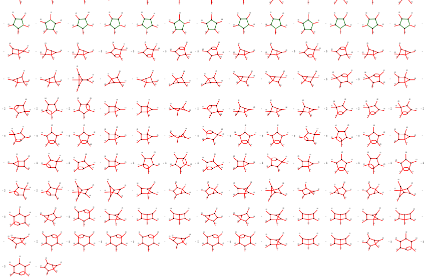
66 terms



Landau Gauge: Five-Gluon Vertex

434 terms





More Fields and Interactions

Now imagine you have 3 fields and 11 interactions.

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

Yang-Mills theory in the maximally Abelian gauge

- 3 fields: diagonal gluons A, off-diagonal gluons B, ghosts c
- 4 three-point interactions: ABB, Acc; BBB, Bcc
- 7 four-point interactions: $AABB$, $AAcc$, $BBBB$, $BBcc$, $cccc$; $ABBB$, $ABcc$

If you have only one field, every type of graph appears once.
If you have several fields, every type of graph can appear in several variations.
$\rightarrow$ Symmetries of the Lagrangian, e. g. ghost-number conservation, restriction in number of diagonal gluons.

The Infrared Solution of the Maximally Abelian Gauge

Non-linear gauge fixing

Minimize off-diagonal components:

$$D_\mu^{ab} B^b = (\delta_{ab}\partial_\mu - g\, f^{abi} A_\mu^i) B^b = 0$$

Scaling solution: power laws for dressing functions in the IR, e. g.

$$\Delta_{\mu\nu}^A = (p^2)^{\delta_A - 1} P_{\mu\nu},$$

with exponents

$-\delta_A = \delta_B = \delta_c \geq 0.$ [R. A., M. Q. H., K. S., in prep.]

$\Rightarrow$ Diagonal gluon relevant degree of freedom = infrared enhanced.

Unique solution for all vertices with even number of legs.

No truncations!

(Landau gauge: ghost $\cong$ relevant d.o.f)

DoDSE: Derivation of Dyson-Schwinger Equations

- Idea: Automatize the derivation process of DSEs.
- Use algorithm explained above in symbolic programming language → *Mathematica*.
- Draw corresponding Feynman diagrams?

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Structures in *DoDSE*

- Fields: list of name and index
- Vertices: V
- Propagators: P
- Bare quantities: S
- Diagrams: op

This is the minimum information needed to write down one epxression unambiguously. Still...

Output of *DoDSE*

```
op[S[{A, i1}, {A, i2}]]

-(1/2) op[S[{A, i1}, {A, i2}, {A, r1}, {A, s1}], P[{A, r1}, {A, s1}]]

-1/2 op[S[{A, i1}, {A, r1}, {A, s1}], V[{A, i2}, {A, t1}, {A, u1}],
  P[{A, r1}, {A, t1}], P[{A, s1}, {A, u1}]]

op[S[{A, i1}, {cb, r1}, {c, s1}], V[{A, i2}, {cb, u1}, {c, t1}],
 P[{c, t1}, {cb, r1}], P[{c, s1}, {cb, u1}]]

-(1/6) op[S[{A, i1}, {A, r1}, {A, r2}, {A, s1}], P[{A, r1}, {A, s2}],
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Fields, indices, propagators, vertices

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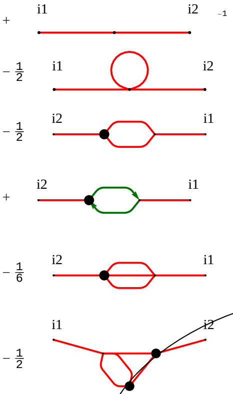


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Fields, indices, propagators, vertices

Basic Input for *DoDSE*

Fields, propagators, interactions, e. g.

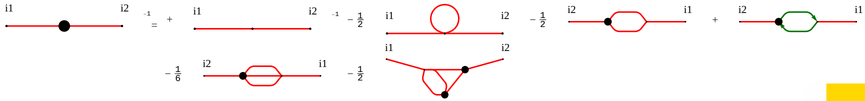
`IA={{A,A},{cb,c},{A,cb,c},{A,A,A},{A,A,A,A}}`

No explicit Lagrangian!

$\Rightarrow$ `AA=doDSE[IA,{A,A}]`

For drawing Feynman graphs: styles for the fields

$\Rightarrow$ `DSEPlot[AA,IA,{{A,Red},{c,Green}}]`



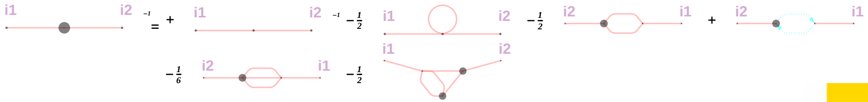
Optional Arguments in *DoDSE*

Some symmetries are taken into account automatically, e. g. Grassmann number conservation.

Sometimes additional information necessary:

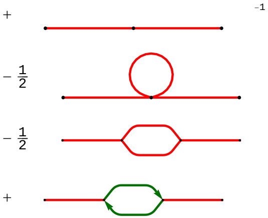
- Restrictions on vertices (e.g. maximally Abelian gauge) $\longrightarrow$ done with simple function.
- Mixed propagators $\longrightarrow$ provide list of allowed possibilities.

For drawing Feynman graphs: options for indices, numerical factors and output format (equation/list)



Outlook: *Symb2Alg*

Symbolic expressions are fine for some tasks, e. g. infrared analysis, but also algebraic expressions are needed!



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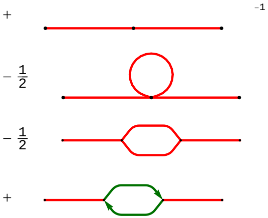

$$\frac{\left(g^{\mu 1\,\nu 1}\,\mathrm{p1}^2 - \mathrm{p1}^{\mu 1}\,\mathrm{p1}^{\nu 1}\right)\,\delta_{\mathrm{a1\,b1}}}{{\mathrm{p1}^2}^2} - \frac{C_A\,g^2\,\left(\mathrm{q1}^{\mu 1}\,\mathrm{q1}^{\nu 1} + 2\,g^{\mu 1\,\nu 1}\,\mathrm{q1}^2\right)\,\delta_{\mathrm{a1\,b1}}}{{\mathrm{q1}^2}^2} - \frac{C_A\,g^2\,\left(\ll 1 \gg\right)\,\delta_{\mathrm{a1\,b1}}}{2\,{\ll 2 \gg^2}^2\,\left(\mathrm{p1}^2 + 2\,\mathrm{p1}\cdot \ll 2 \gg + \mathrm{q1}^2\right)^2} + \frac{C_A\,g^2\,\mathrm{q1}^{\mu 1}\,\left(\mathrm{p1}^{\nu 1} + \mathrm{q1}^{\nu 1}\right)\,\delta_{\mathrm{a1\,b1}}}{\mathrm{q1}^2\,\left(\mathrm{p1}^2 + 2\,\mathrm{p1}\cdot\mathrm{q1} + \mathrm{q1}^2\right)}$$

Mathematica package *Symb2Alg*: Transforms output of *DoDSE* into algebraic expressions.
Depending on Feynman rules compatible with *FeynCalc*.

Summary

- DSEs are important tools in some quantum field theories.
- Derivation of DSEs is straightforward but tedious.
- Simplification in form of diagrammatic rules.
- *DoDSE* implements this algorithm into a *Mathematica* package.

DoDSE provides

- a convenient way to derive DSEs.
- basic possibilities to draw DSEs.
- output that can directly be used in further calculations, e. g. for infrared analysis or to get algebraic expressions that can be manipulated with *FeynCalc* or other functions.