Four-loop Tadpoles and their Applications

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Outline

Applications of tadpoles

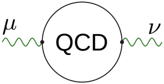
What has been done so far at four loops? (before 2008)

- Master integrals for QED-like single-scale tadpoles
  [Schröder, Vuorinen (2005); Faisst, Tentyukov, Sturm (2006)]
- Decoupling relations for α_s
  [Chetyrkin, Kühn, Sturm; Schröder, Steinhauser (2005)]
- First low energy moment of the vacuum polarization function
  [Chetyrkin, Kühn, Sturm; Boughezal, Czakon, Schutzmeier (2006)]
  determination of m_c and m_b ($e^+e^- \to$ hadrons or lattice data)
  determination of α_s (lattice data)
- Master integrals for the ρ parameter
  [Faisst, PM, Sturm; Boughezal, Czakon (2006)]
- QCD corrections to the ρ parameter
  [Chetyrkin, Kühn, PM, Sturm; Boughezal, Czakon (2006)]
  enters prediction of W mass $\to$ upper bound on SM Higgs mass
- Applications in hot QCD

Example: Quark mass determination from sum rules

The vacuum polarization: correlator of electromagnetic currents j_μ

$$(q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2) = i\int dx\, e^{iqx}\langle 0|Tj_\mu(x)j_\nu(0)|0\rangle =$$



is related to hadron production $R(s) = \dfrac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$

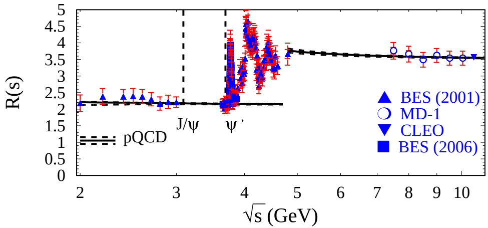
by a dispersion relation $\Pi(q^2) - \Pi(0) = \dfrac{q^2}{12\pi^2}\displaystyle\int ds\, \frac{R(s)}{s(s-q^2)}$.

Compare low q^2 expansion

$$\mathcal{M}_n^{exp} = \int ds\, \frac{R(s)}{s^{n+1}} \quad \text{and} \quad \Pi(q^2) = \frac{3Q_q^2}{16\pi^2}\sum_n C_n \left(\frac{q^2}{4m^2}\right)^n$$

$$\boxed{\Rightarrow m = \frac{1}{2}\left(\frac{9Q_q^2 C_n}{4\mathcal{M}_n^{exp}}\right)^{\frac{1}{2n}}}$$

Features of the sum rule approach



- Reduces the number of scales $\{q^2, m^2\} \to \{m^2\}$ (light quarks are treated as massless) $\longrightarrow$ makes four loop calculations possible
- Long distance effects average out for small n

larger n
- suppresses continuum region
- reduces influence of experimental error
- growing long distance effects
- increasingly difficult to calculate

Status of the vacuum polarization function

- first 8 moments at $\mathcal{O}(\alpha_s^2)$
  [Chetyrkin, Kühn, Steinhauser (1996)]
- moments up to $n = 30$ at $\mathcal{O}(\alpha_s^2)$
  [Boughezal, Czakon, Schutzmeier (2006); Maier, PM, Marquard, (2007)]
- first physical moment at $\mathcal{O}(\alpha_s^3)$
  [Chetyrkin, Kühn, Sturm; Boughezal, Czakon, Schutzmeier (2006)]
- all orders result for n_l^2 at $\mathcal{O}(\alpha_s^3)$
  [Grozin, Sturm (2005)]
- moments up to $n = 30$ for n_f^2 at $\mathcal{O}(\alpha_s^3)$
  [Czakon, Schutzmeier (2007)]

new:
- second physical moment at $\mathcal{O}(\alpha_s^3)$
  [Maier, PM, Marquard (2008)]
- third moment at $\mathcal{O}(\alpha_s^3)$ is under way!

Reduction techniques: Integration-by-Parts

Current multi-loop calculations require up to $\mathcal{O}(10^6)$ Feynman integrals

$\rightarrow$ express integrals as linear combinations of $\mathcal{O}(10)$ master integrals

Basic ingredient: $\boxed{\textbf{Integration-by-Parts relations}}$

[Chetyrkin, Tkachov (1981)]

$$0 = \int d^d k_1 \ldots d^d k_\ell \, \frac{\partial}{\partial k_i^\mu} \, \frac{\{k_j^\mu, p_j^\mu\}}{D_1^{a_1} D_2^{a_2} \ldots D_n^{a_n}}$$

provide relations between integrals
with different propagator powers $\{a_1, \ldots, a_n\}$

Usage: generate and solve system of equations
or construct recursion relations

Reduction techniques: Laporta algorithm

Standard method to solve IBPs: **Laporta algorithm**

- define ordering of integrals
- generate IBPs
- solve systematically for the most difficult integrals by a Gauss elimination-like algorithm

Generally very powerful, but:

- system of equations overdetermined by $\mathcal{O}(3-5)$
- complicated intermediate expressions
- bad combinatorics for large propagator powers; most of the solved integrals are not needed in the calculation

Reduction techniques: Groebner bases

Systematic approach to construct recursion relations:

Groebner bases

- Consider an algebra of operators that shift propagator powers
- Write IBPs as elements of this algebra
- IBPs generate an ideal of relations which are satisfied by Feynman integrals
- Reduce integrals modulo a basis of the ideal
- choose the basis in such a way that the remainder is unique $=$ Groebner basis
- remainder is the reduction formula

Calculates exactly what is needed. Size of intermediate expressions is limited by the number of master integrals.

Reduction techniques: s-bases

How to find a Groebner basis? **Buchberger algorithm**

- guaranteed to construct a basis,
  but may need unlimited resources (CPU, memory)
- practically doesn't succeed even in simple cases

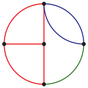
Improvement: **s-bases**

- modified Buchberger algorithm [Smirnov, Smirnov]
- much faster, but not guaranteed to stop
  (strongly dependent on ordering)
- doesn't succeed in all sectors
  $\longrightarrow$ still needs Laporta part
- public implementation available: `FIRE` [A. V. Smirnov]

Reduction of self energy subgraphs

Integrals with self energy insertions have largest propagator powers.

Idea: reduce subgraphs



$=$ 2-loop SE $\times$ 1-loop SE $\times$ connecting propagator
$\times$ cross talking scalar products

Use tensor reduction to remove cross talk
and Laporta algorithm to reduce self energies.

Construct IBPs where the self energies are treated as objects
depending only on their external momentum.

$\rightarrow$ remaining integral is effectively 1-loop!

Very efficient, but limited to special topologies

Combined approaches

Reduction techniques have complementary strengths:

Integrals with highest propagator powers are most difficult for Laporta algorithm (because of combinatorics).

s-basis algorithm happens to fail preferably in lowest sectors.

can be easily done by self energy reduction
$\longrightarrow$ use as plugin for `FIRE`

Major part is done by Laporta algorithm,
keep the system of IBPs as small as possible
and calculate the rest with `FIRE` and self energy reduction.

Second moment of the photon polarization

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi}\bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}_n^{(3)} + \cdots$$

$$\bar{C}_n^{(3)} = C_F T_F^2 n_l^2 \bar{C}^{(3)}_{ll,n} + C_F T_F^2 n_h^2 \bar{C}^{(3)}_{hh,n} + C_F T_F^2 n_l n_h \bar{C}^{(3)}_{lh,n} + C_F T_F n_l \left(C_A \bar{C}^{(3)}_{lNA,n} + C_F \bar{C}^{(3)}_{lA,n}\right) + \bar{C}^{(3)}_{n_f^0,n} + C_F T_F n_h \left(C_A \bar{C}^{(3)}_{hNA,n} + C_F \bar{C}^{(3)}_{hA,n}\right)$$

$$\begin{aligned}
\bar{C}^{(3)}_{n_f^0,2} &= +\frac{64985074258811347}{353072079360000} - \frac{2900811008}{3648645}a_5 - \frac{1662518706713}{21016195200}b_2 + \frac{362601376}{54729675}\log^5 2 - \frac{725202752}{10945935}\zeta_2\log^3 2 \\
&\quad - \frac{1684950406}{3648645}\zeta_4\log 2 + \frac{112680551036302633}{47076277248000}\zeta_3 - \frac{26401638588211}{28021593600}\zeta_4 - \frac{164928917}{270270}\zeta_5 \\
\bar{C}^{(3)}_{hNA,2} &= -\frac{20427854209619}{5649153269760} - \frac{31595849}{11612160}b_2 - \frac{29638030087837}{697426329600}\zeta_3 + \frac{968787977}{15482880}\zeta_4 + \frac{362}{63}\zeta_5 \\
\bar{C}^{(3)}_{lNA,2} &= -\frac{22559166733}{16796160000} - \frac{520999}{4354560}b_2 - \frac{309132631}{12902400}\zeta_3 + \frac{167529079}{5806080}\zeta_4 \\
\bar{C}^{(3)}_{hA,2} &= -\frac{37320009196157}{271593907200} - \frac{130387543}{2177280}b_2 - \frac{5811074101069}{6706022400}\zeta_3 + \frac{2218910663}{1451520}\zeta_4 \\
\bar{C}^{(3)}_{lA,2} &= +\frac{357543003871}{11757312000} + \frac{520999}{2177280}b_2 - \frac{36896356307}{174182400}\zeta_3 + \frac{598455689}{2903040}\zeta_4 \\
\bar{C}^{(3)}_{lh,2} &= +\frac{95040709}{62705664} - \frac{2029}{41472}b_2 - \frac{12159109}{4644864}\zeta_3 + \frac{99421}{55296}\zeta_4\,, \quad \bar{C}^{(3)}_{hh,2} = +\frac{1842464707}{646652160} - \frac{2744471}{1064448}\zeta_3 \\
\bar{C}^{(3)}_{ll,2} &= +\frac{15441973}{19136250} - \frac{32}{45}\zeta_3\,, \quad \bar{C}^{(3)}_{S,2} = +\frac{5881974201847}{8369115955200} + \frac{97011619}{696729600}b_2 + \frac{796232393699}{371960709120}\zeta_3 - \frac{745372259}{185794560}\zeta_4
\end{aligned}$$

$$b_2 = 12\,a_4 + \log^4 2 - 6\zeta_2\log^2 2; \qquad a_n = Li_n(1/2)$$

Impact of recent progess

Reconstruction full q^2 dependence of $\Pi(q^2)$ via Padé approximations [Hoang, Mateu, Zebarjad (2008)]
aims at a contour improved analysis for quark mass determinations

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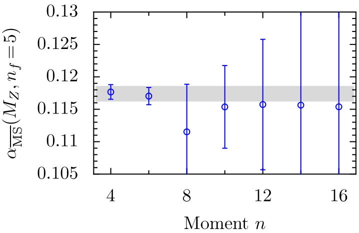
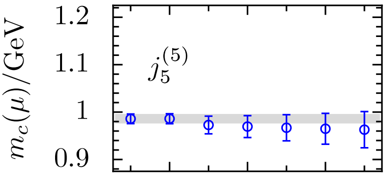
Calculate moments on the lattice (pseudoscalar current correlator)
$\longrightarrow\ \alpha_s,\ m_c$ [HPQCD collaboration & Karlsruhe]





$\boxed{\alpha_s(M_Z) = 0.1174(12)}$ $\boxed{m_c(3\,\mathrm{GeV}) = 0.986(10)\,\mathrm{GeV}}$

Quark masses from $R(s)$

n	$m_b(10\text{ GeV})$	exp	α_s	μ	total	$\delta\bar{C}_n^{(30)}$	$m_b(m_b)$
1	3.593	0.020	0.007	0.002	0.021	—	4.149
2	3.607	0.014	0.012	0.003	0.019	—	4.162
3	3.618	0.010	0.014	0.006	0.019	0.008	4.173
4	3.631	0.008	0.015	0.021	0.027	0.012	4.185

$n = 1$:

$\boxed{m_c(3\,\text{GeV}) = 0.986(13)\,\text{GeV}}$ don't use $n = 1$ for m_b!

$n = 2$: without and with $C_2^{(3)}$

$m_c(3\,\text{GeV}) = 0.979(22)\,\text{GeV}$ $m_b(10\,\text{GeV}) = 3.609(25)\,\text{GeV}$

$\boxed{m_c(3\,\text{GeV}) = 0.976(16)\,\text{GeV}}$ $\boxed{m_b(10\,\text{GeV}) = 3.607(19)\,\text{GeV}}$

Most precise m_b determination available

Conclusion

- Recent progress in reduction of Feynman integrals, inspired by Groebner basis
- Combining different techniques allows for calculations, where Laporta alone fails
- Second moment of current correlators calculated, third moment will be published soon
- Used for precise determinations of m_c, m_b and α_s