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Twisted Poincaré Symmetry of Noncommutative Quantum Field Theory

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• Standard space-time = a manifold $\mathcal{M}$;
points $x \in \mathcal{M} \leftrightarrow$ finite number of real coordinates $x^\mu \in R^4$.

• Usual quantum mechanics:

$$\begin{aligned} \left[x_i, x_j\right] &= 0 \ , \ \left[p_i, p_j\right] = 0 \ , \\ \left[x_i, p_j\right] &= i\hbar\delta_{ij} \ . \end{aligned}$$

• This picture of space-time is likely to break down at very short distances $\sim$ Planck length $\lambda_P \approx 1.6 \times 10^{-33}\,cm$.

• A possible approach to description of physics at short distances is QFT on a NC space-time

• The generalization of commutation relations for the canonical operators of the type

$$x^\mu \to \widehat{x}^\mu \ \ : \quad [\widehat{x}^\mu, \widehat{x}^\nu] \neq 0 \ ,$$

was suggested long ago, in particular, by

Snyder (1947); Heisenberg (1954);
Gol'fand (1962)

• The first physical application: particle noncommutativity in the lowest Landau level

Peierls (1933)

– Point particle moving on a plane (x, y) with external magnetic field B perpendicular to the plane

$$L = \frac{1}{2}mv^2 + \frac{e}{c}\vec{v} \cdot \vec{A} - V \quad \text{with} \quad \vec{A} = (0, Bx)$$

– Set m to zero (strong magnetic field)

$$L_0 = \frac{eB}{c}x\dot{y} - V(x, y)$$

which is of the form $p\dot{q} - h(p, q) \Rightarrow \left(\frac{eB}{c}x, y\right)$ form a canonical pair, i.e.

$$\{x, y\}_{PB} = \frac{c}{eB}$$

– Upon quantization

$$[\hat{x}, \hat{y}] = -i\hbar\frac{c}{eB}$$

$\Rightarrow$

induced noncommutativity of coordinates!

• <u>Practical motivation</u>: the *hope* that QFTs in NC space-time have an improved UV-behaviour.

Snyder (1947)
Grosse, Klimčik and Prešnajder (1996)
Filk(1996)
Chaichian, Demichev and Prešnajder (1998)

• <u>Physical motivations</u>:
- Black hole formation in the process of measurement at small distances ($\sim \lambda_P$) $\Rightarrow$ additional uncertainty relations for **coordinates**

Doplicher, Fredenhagen and Roberts (1994)

- Open string + D-brane theory in the background with antisymmetric tensor

Ardalan, Arfaei and Sheikh-Jabbari (1998)

Seiberg and Witten (1999)

- boundary conditions for open string in constant B-field background:

$$\left[g_{mn}(\partial - \bar{\partial})X^n + 2\pi\alpha' B_{mn}(\partial + \bar{\partial})X^n|\right]_{z=\bar{z}} = 0$$

- corresponding propagator

$$\begin{aligned} < X^m(z,\bar{z})X^n(w,\bar{w}) > &= -\alpha'(g^{mn}log|z-w| - g^{mn}log|z-\bar{w}| \\ &+ G^{mn}log|z-\bar{w}|^2 + \frac{1}{2\pi\alpha'}\theta^{mn}log(-\frac{z-\bar{w}}{\bar{z}-w}) \end{aligned}$$

- in the limit when both z and w approach the real axis: $z = \bar{z} \to \tau_1$, $w = \bar{w} \to \tau_2$, the propagator becomes:

$$< X^m(\tau_1)X^n(\tau_2) >= -\alpha' G^{mn}log(\tau_1 - \tau_2)^2 + \frac{i}{2}\theta^{mn}sign(\tau_1 - \tau_2)$$

implying the commutation relation:

$$[X^m, X^n] = i\theta^{mn},$$
$$\theta^{\mu\nu} = -(2\pi\alpha')^2\left(\tfrac{1}{g+2\pi\alpha' B}B\tfrac{1}{g-2\pi\alpha' B}\right)^{\mu\nu}$$

- *Induced noncommutativity?* See Gravitational and gauge anomalies

Álvarez-Gaumé and Witten (1984)

Green and Schwarz (1984)

NC space-time and field theory; $\star$-product

Heisenberg-like commutation relations

$$[\hat{X}^\mu, \hat{X}^\nu] \; = \; i\theta^{\mu\nu} \; ,$$

$\theta^{\mu\nu}$ - *constant* antisymmetric matrix $\Longrightarrow$ Lorentz invariance violated

$$\textbf{QFT} \;\rightarrow\; \textbf{NC-QFT}: \quad \Phi(x) \;\rightarrow\; \hat{\Phi}(\hat{X}) \; .$$

$$S^{(cl)}[\Phi] \; = \; \int d^4x \left[\frac{1}{2}(\partial^\mu\Phi)(\partial_\mu\Phi) - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4\right] \; ,$$

$$\Downarrow$$

$$S^{(\theta)}[\hat{\Phi}] \; = \; \mathrm{Tr}\left[\frac{1}{2}(\hat{\partial}^\mu\hat{\Phi})(\hat{\partial}_\mu\hat{\Phi}) \; - \frac{1}{2}m^2\hat{\Phi}^2 - \frac{\lambda}{4!}\hat{\Phi}^4\right] \; .$$

Field theory formulation be based on operator (e.g. Weyl) symbols
$\Phi(x)$ = functions on the **commutative** counterpart of the space-time

Weyl-Moyal correspondence

$$\hat{\Phi}(\hat{X}) \longleftrightarrow \Phi(x)$$

$$\hat{\Phi}(\hat{X}) = \int e^{i\alpha\hat{X}}\phi(\alpha)d\alpha, \quad \Phi(x) = \int e^{i\alpha x}\phi(\alpha)d\alpha,$$

where α and x are real variables. Then, using the Baker-Campbell-Hausdorff formula:

$$\hat{\Phi}(\hat{X})\hat{\Psi}(\hat{X}) = \int e^{i\alpha\hat{X}}\phi(\alpha)e^{i\beta\hat{X}}\psi(\beta)d\alpha d\beta = \int e^{i(\alpha+\beta)\hat{X}-\frac{1}{2}\alpha_\mu\beta_\nu[\hat{X}_\mu,\hat{X}_\nu]}\phi(\alpha)\psi(\beta)$$

Hence the **Moyal $\star$-product** is defined:

$$\hat{\Phi}(\hat{X})\hat{\Psi}(\hat{X}) \longleftrightarrow (\Phi \star \Psi)(x),$$

$$(\Phi \star \Psi)(x) \equiv \left[\Phi(x)e^{\frac{i}{2}\theta_{\mu\nu}\frac{\overleftarrow{\partial}}{\partial x_\mu}\frac{\overrightarrow{\partial}}{\partial y_\nu}}\Psi(y)\right]_{x=y}.$$

Thus, all the multiplications (e.g. in the Lagrangian) must be replaced by the $\star$-product

$$S^\theta[\Phi] = \int d^4x \left[\frac{1}{2}(\partial^\mu\Phi)\star(\partial_\mu\Phi) - \frac{1}{2}m^2\Phi\star\Phi - \frac{\lambda}{4!}\Phi\star\Phi\star\Phi\star\Phi\right]$$

UV behaviour of NC QFT and topology

Filk (1996)

Chaichian, Demichev and Prešnajder (1998)

a) *Noncommutative (Moyal) plane*

$$[x_\mu, x_\nu] = i\theta\epsilon_{\mu\nu}, \quad \mu, \nu = 0, 1$$

$$S^\theta_{free}[\Phi] = \int d^2x[(\partial^\mu\Phi^\dagger) \star (\partial_\mu\Phi) - m^2\Phi^\dagger \star \Phi]$$

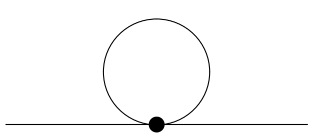
The propagator is the same as in the commutative theory:

$$G^\theta(x,y) = \frac{1}{(2\pi)^2}\int d^2k\, \frac{\imath k(x-y)}{k^2 - m^2}$$

If the interaction is switched on, the action

$$S^\theta_{int} = \frac{\lambda}{4!}\int d^2x\Big(\Phi^\dagger \star \Phi \star \Phi^\dagger \star \Phi\Big)$$

produces vertices containing factors proportional to $\theta k^2/2$, plus additional phase factors $exp[\pm i(k_1 \times k_2 + k_3 \times k_4)/2]$ (here, $k \times p = \theta_{\mu\nu}k^\mu p^\nu$) $\Rightarrow$



is UV-**divergent**

b) *Compact Lie algebra* $su(2)$

- Lie-algebra type commutation relations

$$[\hat{x}_i, \hat{x}_j] = \imath\lambda\epsilon_{ijk}\hat{x}_k \ ,$$

$$\sum_{i=1}^{3} \hat{x}_i^2 = \lambda^2 s(s+1) \ , \quad s = 0, 1/2, 1, ...$$

- "**fuzzy sphere**"

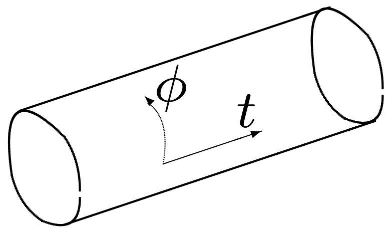
Berezin (1975)
Madore (1991)
Grosse, Prešnajder et al. (1996), (1997)

All irreducible representations are finite dim. $\Rightarrow$ for a fixed $s < \infty$:

$$\hat{\phi}(\hat{x}_i) = [(2s+1) \times (2s+1)] \ - \ \text{matrices}$$

$\Rightarrow$ Any calculation for $S^{(NC)}$ with $s < \infty$ reduces to manipulations with **finite-dimensional** matrices $\Rightarrow$ **no UV-divergences**.

c) *Non-compact Lie algebra*

2-dim. cylinder: 

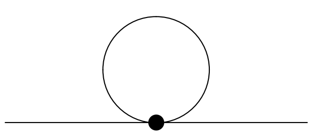
$$C_\rho = \{(x_\pm, t),\ t \in \mathbb{R},\ x_\pm = \rho \pm \imath\phi,\ \rho = \text{const}\}$$

Poisson brackets: $\{t, x_\pm\}_p = \pm\imath x_\pm$, $\quad \{x_+, x_-\}_p = 0$

$x_+ x_- = \rho^2$ is a central element.

$$S^{(cl)}[\varphi, \varphi^*] = \int_{C_\rho} d^2x \Big[\varphi^* \big(\Box + m^2\big)\varphi + \frac{1}{2}(\varphi^*\varphi)^2\Big]$$

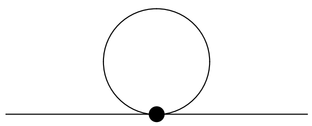
$$\Box\varphi = \{t, \{t, \varphi\}_p\}_p + \rho^2 \{x_+, \{x_-, \varphi\}_p\}_p$$

 $$\sim \quad g G_0^{(cl)}(x, t; x, t) = \sum_{k=0}^{\infty} \frac{g}{\sqrt{k^2+m^2}} = \infty$$

Space-time quantization:

$$\{\cdot,\cdot\}_p \to (\lambda/\imath)[\cdot,\cdot], \quad \int d^2x \to \mathrm{Tr}$$

$\Rightarrow$



$$\sim \quad g\int_0^{\pi/\lambda} d\omega\, \frac{\cot\left(\sqrt{\Omega_\lambda^2(\omega)-m^2+\imath\varepsilon}\right)}{\sqrt{\Omega_\lambda^2(\omega)-m^2+\imath\varepsilon}}, \qquad \Omega_\lambda(\omega) \equiv \frac{2}{\lambda}\sin\frac{\omega\lambda}{2}$$

convergent

$\Rightarrow$ UV-finiteness for **planar** diagrams (in sharp distinction from the case of a flat NC-space-time)!! As usual, nonplanar diagrams have even better convergence properties.

In conclusion, *global topological properties are crucial*. In order to achieve the removal of UV divergences of a QFT formulated in NC space-time of arbitrary dimension, at most one dimension (e.g., time) is allowed to be non-compact.

Chaichian, Demichev and Prešnajder (1998)

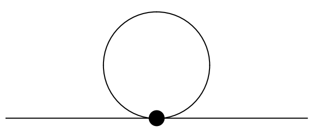
UV/IR mixing

Minwalla, Van Raamsdonk and Seiberg (1999)

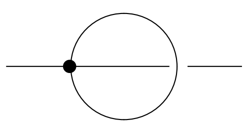
ϕ^4 theory with Euclidean action:

$$S = \int d^4x \Big(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} g^2 \phi \star \phi \star \phi \star \phi \Big)$$

Consider the 1-loop corrections to the two-point function $\Gamma^{(2)}$, coming from two diagrams:



$$\Gamma^{(2)}_{planar} = \frac{g^2}{3(2\pi)^4} \int \frac{d^4k}{k^2+m^2} \approx \frac{g^2}{48\pi^2} \Big(\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} + \cdots \Big)$$



$$\Gamma^{(2)}_{nonplanar} = \frac{g^2}{6(2\pi)^4} \int \frac{d^4k}{k^2+m^2} e^{ik\times p} \approx \frac{g^2}{96\pi^2} \Big(\Lambda^2_{eff} - m^2 \ln \frac{\Lambda^2_{eff}}{m^2} + \cdots \Big)$$

where $\Lambda^2_{eff} = \frac{1}{1/\Lambda^2 + p \circ p}$, $\quad p \circ q \equiv -p_\mu \theta^2_{\mu\nu} q_\nu = |p_\mu \theta^2_{\mu\nu} q_\nu|$.

If $|p| \ll \frac{1}{\Lambda\theta}$, then $\Lambda_{eff} \approx \Lambda$. If $\frac{1}{\Lambda\theta} \ll |p|$, then $\Lambda_{eff} \approx \frac{1}{|p_\mu \theta^{\mu\nu}|}$.

Renormalize at fixed p, θ; subtract planar mass divergence. In the $\Lambda \to \infty$ limit:

$$\Gamma^{(2)}(p) = p^2 + M^2 + \frac{g^2}{96\pi^2 p \circ p} - \frac{g^2 M^2}{96\pi^2} \ln \frac{1}{M^2 p \circ p} \ .$$

An UV divergence has turned into an IR one. The Green's functions will have singularities in $|p \cdot \theta|$.

• At fixed p, the $\theta \to 0$ limit is singular, non-analytic;

• At fixed θ, new IR singularities appear;

• The limits $\theta \to 0$ and $\Lambda \to \infty$ do not commute (θ and $\hbar$ quantizations do not commute).

• An analysis of UV/IR mixing in NC $U(n)$ theories suggests that the closed string tachyon that couples non-trivially to the brane (in contrast to the commutative case) is behind the instabilities in field theory.

Armoni and Lopez (2001)

Unitarity

For on-shell matrix elements, unitarity implies that

$$2\mathrm{Im} M_{ba} = \sum_n M_{bn} M_{na},$$

where M_{ba} is the transition matrix element between the states a and b.

NC ϕ^3 theory at one loop

Gomis and Mehen (2000)

- Cutting rules

• Evaluate sum over final states $\sum |M|^2 = \frac{\lambda^2}{16\pi} \frac{\sin(\gamma\sqrt{p^2\, p\circ p}/2)}{\sqrt{p^2\, p\circ p}}, \gamma = \sqrt{1 - 4\frac{m^2}{p^2}}$

• For $\theta^2_{\mu\nu} > 0$, Im $M = \frac{\lambda^2}{32\pi} \frac{\sin(\gamma\sqrt{p^2\, p\circ p}/2)}{\sqrt{p^2\, p\circ p}}$,
space-space noncommutativity, unitarity fulfilled

• For $\theta^2_{\mu\nu} < 0$, Im $M = \frac{\lambda^2}{64\pi} \int_0^1 dx\, J_0(\sqrt{|p\circ p|(m^2 + |p^2|x(1-x))})$
space-time noncommutativity, unitarity violated

• Notice that for $\theta^2_{\mu\nu} > 0$, i.e. space-space noncommutativity, the unitarity condition is fulfilled, while for $\theta^2_{\mu\nu} < 0$, i.e. space-time noncommutativity, unitarity is violated.

• It was also shown that theories with light-like noncommutativity $\theta^2 = \theta_{\mu\nu}\theta^{\mu\nu} = 0$, i.e. $\theta^{0i} = -\theta^{1i}$ (the remaining $\theta^{ij} = 0$), are unitary.

Aharony, Gomis and Mehen (2000)

• Notice: those noncommutative field theories which are unitary can be obtained as decoupled field theory limits of string theory, while those which are not unitary can not be obtained from string theory.

• Attempts to prove the unitarity of field theories with space-time noncommutativity, in the Hamiltonian approach:

Bahns, Doplicher, Fredenhagen and Piacitelli (2002)

Liao and Sibold (2002)

Causality

Scattering in noncommutative field theory

Seiberg, Susskind and Toumbas (2000)

• Consider 2+1 dimensional NC ϕ^4 theory in lowest order perturbation theory ($2 \to 2$ particle scattering).

• In the case of space-space noncommutativity ($\theta_{0i} = 0$)

$$\psi_{out}(y) \approx \psi_{in}(y)\delta(y - \frac{1}{2}\theta P_x),$$

i.e., the outgoing scattered wave appears to originate from the displaced position $y = \theta P_x/2$; **causality is preserved**.

• In the case of space-time noncommutativity $\theta_{0i} \neq 0$, choose for the incoming wavepacket

$$\phi_{in}(p) \sim E_p \left(e^{-\frac{(p-p_0)^2}{\lambda}} + e^{-\frac{(p+p_0)^2}{\lambda}} \right) .$$

Then

$$\begin{aligned} \phi_{out}(x) &\sim g \left[F(x;-\theta) + 4\sqrt{\lambda} e^{-\lambda \frac{x^2}{4}} e^{ip_0 x} + F(x;\theta) \right] \\ &+ (p_0 \to -p_0) \ , \end{aligned}$$

where

$$F(x;\theta) \equiv \frac{1}{\sqrt{-4i\theta}} e^{-\frac{(x+8p_0\theta)^2}{64\theta^2\lambda}} e^{-i\frac{(x-\frac{p_0}{2\lambda^2\theta})^2}{16\theta}} e^{i\frac{p_0^2}{4\lambda^2\theta}}.$$

• The outgoing packet splits into three parts, one of them advanced and one retarded. The advanced packet appears to leave the collision place at a time $t = 8p_0\theta$ before the incoming packet arrived. The **acausal** effect increases with the energy.

Space-time symmetry of NC QFT

• $\theta_{\mu\nu}$ antisymmetric *constant* matrix $\Rightarrow$ **Lorentz invariance violated** (for a dimension of space-time $D > 2$).

• Translational invariance preserved.

• On 4-dimensional space there exists a frame in which the antisymmetric matrix $\theta_{\mu\nu}$ takes the form:

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta' \\ 0 & 0 & -\theta' & 0 \end{pmatrix}.$$

Lorentz group broken to $SO(1,1) \times SO(2)$ subgroup.

Álvarez-Gaumé, Barbón and Zwicky (2001)

• Problem with the representations: both $SO(1,1)$ and $SO(2)$ being Abelian groups, they have only one-dimensional unitary irreducible representation and thus no spinor, vector etc. representations!

Twisted Poincaré symmetry

Chaichian, Kulish, Nishijima and Tureanu (2004)

Chaichian, Prešnajder and Tureanu (2004)

• Action of NC QFT written with $\star$-product, though it violates Lorentz symmetry, it is invariant under the twisted Poincaré algebra

• Deform the universal enveloping of the Poincaré algebra $\mathcal{U}(\mathcal{P})$ with Abelian twist element $\mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$

Drinfeld (1983)

Reshetikhin (1990)

$$\mathcal{F} = exp\left(\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right)$$

• Commutation relations of Poincaré generators not changed:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \ , \\
[M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu) \ , \\
[M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} - \eta_{\nu\alpha}M_{\mu\beta} + \eta_{\nu\beta}M_{\mu\alpha})
\end{aligned}$$

Essential physical implication: the representations of the twisted Poincaré algebra are the same as the ones of usual Poincaré algebra

• The twist deforms the action of $\mathcal{U}(\mathcal{P})$ in the tensor product of representations, defined by the coproduct

$$\Delta_0 : \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P}) \,, \quad \Delta_0(Y) = Y \otimes 1 + 1 \otimes Y \ ,$$

$$\Delta_0(Y) \mapsto \Delta_t(Y) = \mathcal{F} \Delta_0(Y) \mathcal{F}^{-1}$$

Namely the coproduct of the Lorentz algebra generators is changed:

$$\Delta_t(M_{\mu\nu}) = e^{\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta} \Delta_0(M_{\mu\nu}) e^{-\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta} \ .$$

• The twist also deforms the multiplication in the algebra of representation of the Poincaré algebra, i.e. algebra of fields $\mathcal{A}_\theta$:

$$m_t(\phi(x) \otimes \psi(x)) = m \circ \mathcal{F}^{-1}(\phi(x) \otimes \psi(x)) =: \phi(x) \star \psi(x)$$

i.e., with the realization on Minkowski space $P_\mu = i\partial_\mu$

$$\begin{aligned} \phi(x) \star \psi(x) &= m \circ e^{-\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} (\phi(x) \otimes \psi(x)) = m \circ e^{\frac{i}{2}\theta^{\mu\nu} \partial_\mu \otimes \partial_\nu} (\phi(x) \otimes \psi(x)) \\ &= \phi(x) e^{\frac{i}{2}\theta^{\mu\nu} \overleftarrow{\partial_\mu} \overrightarrow{\partial_\nu}} \psi(x) \end{aligned}$$

• The twisted Poincaré symmetry exists provided that, in a Lagrangean: (i) we consider $\star$-products among functions instead of the usual one and (ii) we take the proper action of generators specified by the twisted coproduct.

• As a byproduct with major physical implications, the representation content of NC QFT, invariant under the twist-deformed Poincaré algebra, is identical to the one of the corresponding commutative theory with usual Poincaré symmetry $\Rightarrow$ representations (fields) are classified according to their MASS and SPIN.

• New concept of relativistic invariance: while symmetry under usual Lorentz transformations guarantees the relativistic invariance of a theory, in NC QFT the concept of relativistic invariance should be replaced by the requirement of invariance of the theory under twisted Poincaré transformations.

Precursors

-in the context of NC string theory, using $\mathcal{R}$-matrix

Watts (1999)

- mostly in the context of braided field theory, using the dual language of Hopf algebras

Oeckl (2000)

Developments

- differential calculus, twisted diffeomorphisms and NC gravity

Wess (2004)
Aschieri, Blohmann, Dimitrijevic, Meyer, Schupp and Wess (2005)
Aschieri, Dimitrijevic, Meyer and Wess (2005)
Álvarez-Gaumé, Meyer and Vázquez-Mozo (2006)

- twist, spin-statistics and NC gravity

Balachandran et al. (2005), (2006), (2007)
Szabo (2006), Riccardi and Szabo (2007)

- supersymmetric twisted Poincaré algebra

Kobayashi and Sasaki (2005)
Zupnik (2005)
Ihl and Saemann (2005)

- global counterpart of the twisted Poincaré algebra

Gonera, Kosinski, Maslanka and Giller (2005)

...

Some known implications...

- **$\mathcal{R}$-matrix and new concept of permutations**

- $\mathcal{R}$-matrix relates Δ_t and $\Delta_t^{op} = \tau \circ \Delta_t$, where τ is the flip operator:

$$\mathcal{R}\Delta_t = \Delta_t^{op}\mathcal{R}, \quad \mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 \Rightarrow \mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} = exp(-i\theta^{\mu\nu} P_\mu \otimes P_\nu)$$

- Concept of permutation changes

Chari and Pressley (book 1994)
Chaichian and Demichev (book 1996)
Fiore and Schupp (1995)
Kulish and Mudrov (2004)

$$\tau \to \tau(\mathcal{R}) = \mathcal{F}\tau\mathcal{F}^{-1} = \tau\ \mathcal{R}$$

-in NC QFT, consider realization of P_μ as quantum momentum operator

$$P_\mu = \int d^3k\, k_\mu\, a^\dagger(k)a(k), \quad [P_\mu, a(k)] = -k_\mu a(k), \quad [P_\mu, a^\dagger(k)] = k_\mu a^\dagger(k)$$

- $\star$-product between creation and annihilation operators, e.g.

$$a^\dagger(k) \star a^\dagger(p) = m \circ \mathcal{F}^{-1}\left(a^\dagger(k) \otimes a^\dagger(p)\right) = a^\dagger(k)a^\dagger(p)e^{-\frac{i}{2}k_\mu\theta^{\mu\nu}p_\mu}$$

$$\Rightarrow a^\dagger(k) \star a^\dagger(p) = a^\dagger(p) \star a^\dagger(k)e^{-ik_\mu\theta^{\mu\nu}p_\mu}$$

- but $\mu \circ \mathcal{F}^{-1}\tau(\mathcal{R})\left(a^\dagger(k) \otimes a^\dagger(p)\right) = a^\dagger(p) \star a^\dagger(k)e^{-ik_\mu\theta^{\mu\nu}p_\mu}$

$\Rightarrow$ statistics OK, shown also directly in

Bu, Kim, Lee, Vac and Yee (2006)

- **Twisted tensor product of two copies of $\mathcal{A}_\theta$**

$$(a_1 \otimes 1)(1 \otimes a_2) = a_1 \otimes a_2, \quad \text{but } (1 \otimes a_2)(a_1 \otimes 1) = (\mathcal{R}_2 a_1) \otimes (\mathcal{R}_1 a_2),$$

$$a_1, a_2 \in \mathcal{A}_\theta$$

$$\Rightarrow \qquad x^\mu y^\nu - y^\nu x^\mu := (x^\mu \otimes 1)(1 \otimes y^\nu) - (1 \otimes y^\nu)(x^\mu \otimes 1)$$
$$= (x^\mu \otimes x^\nu) - (\mathcal{R}_2 x^\mu) \otimes (\mathcal{R}_1 y^\nu) = (x^\mu \otimes x^\nu) - (x^\mu \otimes x^\nu) + i\theta^{\mu\nu}$$

$$\Rightarrow \phi(x) \star \phi(y)$$

Oeckl (2000)

Kulish (2005)

- **Global counterpart of twisted Poincaré algebra**

Oeckl (2000)

Gonera, Kosinski, Maslanka and Giller (2005)

- parameters $\Lambda^\mu{}_\nu, a^\mu$ of global Poincaré transformations generate the algebra dual to $\mathcal{U}(P)$

$$x^\mu \to \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1$$

- parameters of finite translations do not commute $\Rightarrow$ NONLOCALITY

$$[a^\mu, a^\nu] = i\theta^{\mu\nu} - i\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \theta^{\alpha\beta}$$
$$[\Lambda^\mu{}_\nu, a^\mu] = [\Lambda^\mu{}_\alpha, \Lambda^\nu{}_\beta] = 0$$

Is the concept of twist a symmetry principle in constructing NC field theories, i.e. any symmetry that NC field theories may enjoy, be it space-time or internal symmetry, global or local, should be formulated as a twisted symmetry?

Twisted gauge symmetry?

- NC gauge theories - traditional approach

Hayakawa (1999)

The NC QED action:

$$S_{NC\ QED} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \bar{\Psi} \star (\not{D} - m)\Psi + L_{gauge} + L_{ghost} \right)$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu \star A_\nu - A_\nu \star A_\mu) \ , \\ D_\mu \Psi &= \partial_\mu \Psi - i A_\mu \star \Psi \ . \end{aligned}$$

NC gauge group elements:

$$\begin{aligned} &U(x) = \exp_\star\{i\lambda\} \equiv 1 + i\lambda - \frac{1}{2}\lambda \star \lambda \ + \ \ldots. \ , \\ &U(x) \star U(x)^{-1} = U(x)^{-1} \star U(x) = 1 \ . \end{aligned}$$

Gauge transformations:

$$\begin{aligned} &A_\mu \to A'_\mu(x) = U(x) \star A_\mu \star U^{-1}(x) + iU(x) \star \partial_\mu U(x)^{-1} \ , \\ &\Psi(x) \to \Psi'(x) = U(x) \star \Psi(x) \ . \end{aligned}$$

- Remark: only NC $U(n)$ groups close (not, e.g., $SU(n)$)
- **No-go theorem**

Terashima (2000)

Chaichian, Prešnajder, Sheikh-Jabbari and Tureanu (2001)

(i) the local NC $u(n)$ algebra only admits the irreducible $n \times n$ matrix-representation. Hence the gauge fields are in the $n \times n$ matrix form, while the matter fields *can only be* in fundamental, adjoint or singlet states;

(ii) for any NC gauge group consisting of several simple-group factors, the matter fields can transform nontrivially under *at most* two group factors.

- Applications:

- NC Standard Model

Chaichian, Prešnajder, Sheikh-Jabbari and Tureanu (2001)
Chaichian, Kobakhidze and Tureanu (2004)
Khoze and Levell (2004)

- NC MSSM

Arai, Saxell and Tureanu (2006)

• Attempt to *twist gauge transformations*: extend the Poincaré algebra by semidirect product with the gauge generators and apply the Abelian twist $\mathcal{F} = e^{\left(\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right)}$ also to the coproduct of the gauge generators

Vassilevich (2006)

Aschieri, Dimitrijevic, Meyer, Schraml and Wess (2006)

- infinitesimal gauge transformation of the individual fields the usual form (without $\star$-product):

$$\delta_\alpha \Phi(x) = \alpha(x)\Phi(x) \ , \quad \alpha(x) = i\alpha^a(x) T_a \ , \quad [T_a, T_b] = i f_{abc} T_c$$

- claim

$$\delta_\alpha(\Phi_1(x) \star \Phi_2(x)) = i\alpha^a(x) \left[(\Phi_1(x) T_a^{(1)}) \star \Phi_2(x) + \Phi_1(x) \star (T_a^{(2)} \Phi_2(x)) \right]$$

- consequences: any gauge algebra would close and any representation is allowed, just as in the commutative case, i.e. contradiction with the no-go theorem!

- Contradiction with the gauge principle:

$$\delta_\alpha(\Phi_1(x)\star\Phi_2(x)) = i\alpha^a(x)\left[(\Phi_1(x)T_a^{(1)})\star\Phi_2(x) + \Phi_1(x)\star(T_a^{(2)}\Phi_2(x))\right].$$

is valid only if one *assumes* that, once $\delta_\alpha\Phi(x) = \alpha(x)\Phi(x)$, then also

$$\delta_\alpha((-i)^n P_{\mu_1}...P_{\mu_n}\Phi(x)) = \delta_\alpha(\partial_{\mu_1}...\partial_{\mu_n}\Phi(x)) = \alpha(x)(\partial_{\mu_1}...\partial_{\mu_n}\Phi(x))$$

which is true only when the "local" parameter α^a is global!

$$\begin{aligned}
\delta_\alpha(\Phi_1\star\Phi_2) &= m_\star\circ\Delta_t(\alpha(x))(\Phi_1(x)\otimes\Phi_2(x)) \\
&= m\circ\mathcal{F}^{-1}\mathcal{F}\Delta_0(\alpha(x))\mathcal{F}^{-1}(\Phi_1(x)\otimes\Phi_2(x)) \\
&= m\circ\Delta_0(\alpha)\mathcal{F}^{-1}(\Phi_1(x)\otimes\Phi_2(x)) \\
&= m\circ\Delta_0(\alpha)e^{\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu\right)}(\Phi_1(x)\otimes\Phi_2(x)) \\
&= m\circ(\delta_\alpha\otimes 1 + 1\otimes\delta_\alpha)\left[\Phi_1\otimes\Phi_2 + \frac{i}{2}\theta^{\mu\nu}(\partial_\mu\Phi_1\otimes\partial_\nu\Phi_2) + \cdots\right.
\end{aligned}$$

Chaichian and Tureanu (2006)

However

$$\delta_\alpha(D_{\mu_1}...D_{\mu_n}\Phi(x)) = \alpha(x)(D_{\mu_1}...D_{\mu_n}\Phi(x))$$

- Non-Abelian twist element of $\mathcal{U}(\mathcal{P} \ltimes \mathcal{G})$:

$$\mathcal{T} = \exp\left(-\frac{i}{2}\theta^{\mu\nu} D_\mu \otimes D_\nu + \mathcal{O}(\theta^2)\right),$$

a power series expansion, such that $\mathcal{T}$ would satisfy the twist conditions:

$$(\mathcal{T} \otimes 1)(\Delta_0 \otimes id)\mathcal{T} = (1 \otimes \mathcal{T})(id \otimes \Delta_0)\mathcal{T}, \quad (\epsilon \otimes id)\mathcal{T} = 1 = (id \otimes \epsilon)\mathcal{T}$$

Chaichian, Tureanu and Zet (2006)

- new $\bigstar$-product

$$\Phi \bigstar \Psi = m \circ \exp\left(\frac{i}{2}\theta^{\mu\nu} D_\mu \otimes D_\nu + \mathcal{O}(\theta^2)\right)(\Phi \otimes \Psi)$$

should reduce to the usual Moyal $\star$-product for ordinary functions on the Minkowski space, which have to be considered in the 1-dimensional (trivial) representation of the gauge group G, i.e. $T_a f(x) = 0$, i.e. $D_\mu f(x) = \partial_\mu f(x) \Longrightarrow [x_\mu, x_\nu]_\bigstar = [x_\mu, x_\nu]_\star = i\theta_{\mu\nu}$

-Possible typical second order terms are (with all permutations):

$$\theta^{\mu\nu}\theta^{\rho\sigma}(1 \otimes D_\mu D_\nu D_\rho D_\sigma) \quad \text{and} \quad \theta^{\mu\nu}\theta^{\rho\sigma}(D_\mu D_\nu D_\rho D_\sigma \otimes 1)$$

$$\theta^{\mu\nu}\theta^{\rho\sigma}(D_\mu \otimes D_\nu D_\rho D_\sigma) \quad \text{and} \quad \theta^{\mu\nu}\theta^{\rho\sigma}(D_\mu D_\nu D_\rho \otimes D_\sigma)$$

$$\theta^{\mu\nu}\theta^{\rho\sigma}(D_\mu D_\nu \otimes D_\rho D_\sigma),$$

- Due to the antisymmetry of $\theta_{\mu\nu}$, the second order in θ, the most general ansatz is

$$
\begin{aligned}
\mathcal{T} \;&=\; \exp\{-\frac{i}{2}\theta^{\mu\nu}\left(D_\mu \otimes D_\nu + 1 \otimes F_{\mu\nu} + F_{\mu\nu} \otimes 1\right) \\
&+\; \frac{1}{2}\left(-\frac{i}{2}\right)^2 \theta^{\mu\nu}\theta^{\rho\sigma}[a\ D_\mu \otimes D_\sigma D_\nu D_\rho + b\ D_\mu \otimes D_\nu D_\sigma D_\rho + c\ D_\mu \otimes D_\sigma D_\rho D_\nu \\
&+\; a'\ D_\sigma D_\nu D_\rho \otimes D_\mu + b'\ D_\nu D_\sigma D_\rho \otimes D_\mu + c'\ D_\sigma D_\rho D_\nu \otimes D_\mu + \mathcal{O}(\theta^2)]\}\,,
\end{aligned}
$$

-requirement to fulfill the twist condition leads to:

$$a = a' = -1, \text{ but } \quad a + a' = 2!$$

$\Rightarrow$ a non-Abelian twist element, which would generalize the Abelian twist in a gauge covariant manner cannot exist, i.e. Poincaré symmetry and internal gauge symmetry cannot be unified under a common twist

- situation is reminiscent of the Coleman-Mandula no-go theorem

COULD SUPERSYMMETRY PROVIDE THE SOLUTION?

Some problems still to be understood and solved:

• The relation between the Seiberg-Witten map and the no-go theorem for NG gauge field theories;

• Analog of Froissart-Martin bound for the cross-section in NC QFT: Jost-Lehmann-Dyson representation, Lehmann analyticity ellipse

$$\sigma_{\mathrm{tot}}(E) \leq \; c \; \ln^2 \frac{E}{E_0}$$

• Dirac quantization condition for magnetic monopole

$$e\mu = \frac{n\hbar}{2} c$$

• Looking really at the solutions of NC Gravity, to find out about the singularity of solutions, Schwarzschild, Reissner-Nordström, black holes... and repeat the same arguments for the consistency of emergence of the noncommutativity of space-time based on QM and the NEW way of black hole formation.

• ...