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Exact Results in Noncommutative Quantum Field Theory: CPT and Spin-Statistics Theorems

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NC space-time and field theory; $\star$-product

Heisenberg-like commutation relations

$$[\hat{X}^\mu, \hat{X}^\nu] \ = \ i\theta^{\mu\nu} \ ,$$

$\theta^{\mu\nu}$ - *constant* antisymmetric matrix $\Rightarrow$ Lorentz invariance violated

$$\textbf{QFT} \ \rightarrow \ \textbf{NC-QFT}: \quad \Phi(x) \ \rightarrow \ \hat{\Phi}(\hat{X}) \ .$$

$$S^{(cl)}[\Phi] \ = \ \int d^4x \left[\frac{1}{2}(\partial^\mu\Phi)(\partial_\mu\Phi) - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4\right] \ ,$$

$$\Downarrow$$

$$S^{(\theta)}[\hat{\Phi}] \ = \ \mathrm{Tr}\left[\frac{1}{2}(\hat{\partial}^\mu\hat{\Phi})(\hat{\partial}_\mu\hat{\Phi}) \ - \frac{1}{2}m^2\hat{\Phi}^2 - \frac{\lambda}{4!}\hat{\Phi}^4\right] \ .$$

Field theory formulation be based on operator (e.g. Weyl) symbols
$\Phi(x)$ = functions on the **commutative** counterpart of the space-time

Weyl-Moyal correspondence

$$\hat{\Phi}(\hat{X}) \longleftrightarrow \Phi(x)$$

$$\hat{\Phi}(\hat{X}) = \int e^{i\alpha\hat{X}}\phi(\alpha)d\alpha, \quad \Phi(x) = \int e^{i\alpha x}\phi(\alpha)d\alpha,$$

where α and x are real variables. Then, using the Baker-Campbell-Hausdorff formula:

$$\hat{\Phi}(\hat{X})\hat{\Psi}(\hat{X}) = \int e^{i\alpha\hat{X}}\phi(\alpha)e^{i\beta\hat{X}}\psi(\beta)d\alpha d\beta = \int e^{i(\alpha+\beta)\hat{X}-\frac{1}{2}\alpha_\mu\beta_\nu[\hat{X}_\mu,\hat{X}_\nu]}\phi(\alpha)\psi(\beta)$$

Hence the **Moyal $\star$-product** is defined:

$$\hat{\Phi}(\hat{X})\hat{\Psi}(\hat{X}) \longleftrightarrow (\Phi \star \Psi)(x),$$

$$(\Phi \star \Psi)(x) \equiv \left[\Phi(x)e^{\frac{i}{2}\theta_{\mu\nu}\frac{\overleftarrow{\partial}}{\partial x_\mu}\frac{\overrightarrow{\partial}}{\partial y_\nu}}\Psi(y)\right]_{x=y}.$$

Thus, all the multiplications (e.g. in the Lagrangian) must be replaced by the $\star$-product

$$S^\theta[\Phi] = \int d^4x\left[\frac{1}{2}(\partial^\mu\Phi)\star(\partial_\mu\Phi) - \frac{1}{2}m^2\Phi\star\Phi - \frac{\lambda}{4!}\Phi\star\Phi\star\Phi\star\Phi\right]$$

CPT symmetry in NC QFT in Hamiltonian approach

Sheikh-Jabbari (2000)

Chaichian, Nishijima, Tureanu (2002)

CPT in NC QED

- **Parity** - conserved in NC QED;

- **Charge conjugation** - violated in NC QED (a theory on $\mathbb{R}^4_\theta$ is transformed into the theory on $\mathbb{R}^4{}_{-\theta}$);

- **Time reversal** - violated in NC QED (again, a theory on $\mathbb{R}^4_\theta$ is transformed into the theory on $\mathbb{R}^4{}_{-\theta}$);

- As a result, NC QED is CP and T violating, but **CPT remains "accidentally" valid**.

Sheikh-Jabbari (2000)

General proof of the CPT theorem for NC fields

Chaichian, Nishijima, Tureanu (2002)

- CPT transformations for *elementary* fields:

$$\psi_\alpha^{CPT}(x) = (i\gamma_5)_{\alpha\beta}\psi_\beta(-x)\ , \quad \bar\psi_\alpha^{CPT}(x) = \bar\psi_\beta(-x)(i\gamma_5)_{\beta\alpha}\ ,$$
$$\phi^{CPT}_{\lambda_1\ldots\lambda_n}(x) = (-1)^n\phi_{\lambda_1\ldots\lambda_n}(-x).$$

- The CPT theorem states that:

$$\mathcal{H}_{int}^{CPT}(x) = \mathcal{H}_{int}(-x)\ .$$

- Take a n-linear form for $\mathcal{H}(x)$:

$$\begin{aligned}\mathcal{H}(x) &= \sum_{i_1\ldots i_n} f_{i_1\ldots i_n}(\Phi^1_{i_1}\star\ldots\star\Phi^n_{i_n})(x)\\ &= e^D\sum_{i_1\ldots i_n} f_{i_1\ldots i_n}\Phi^1_{i_1}(x_1)\ldots\Phi^n_{i_n}(x_n)|_{x_1=\ldots=x_n\equiv x}\ ,\end{aligned}$$

with $D = \frac{i}{2}\theta^{\mu\nu}(\partial_\mu^{x_1}\partial_\nu^{x_2} + \partial_\mu^{x_2}\partial_\nu^{x_3} + \ldots + \partial_\mu^{x_{n-1}}\partial_\nu^{x_n})$
(i_j with $j = 1, \ldots, n$ stand for spinorial or tensorial indices and the coefficients $f_{i_1\ldots i_n}$ are so chosen as to make $\mathcal{H}(x)$ a scalar under proper Lorentz transformations, *in the local limit*).

• The CPT transform of $\mathcal{H}(x)$ is given by:

$$\begin{aligned}\mathcal{H}^{CPT}(x) &= e^D \sum_{i_1\ldots i_n} f_{i_1\ldots i_n}(\Phi^n_{i_n})^{CPT}(x_n)\ldots(\Phi^1_{i_1})^{CPT}(x_1)|_{x_1=\ldots=x_n\equiv x} \\ &= e^D \sum_{i_1\ldots i_n} f'_{i_1\ldots i_n}\Phi^n_{i_n}(-x_n)\ldots\Phi^1_{i_1}(-x_1)|_{x_1=\ldots=x_n\equiv x}\ ,\end{aligned}$$

where f' is given by

$$f'_{i_1\ldots i_n} = (-1)^{F/2} f_{i_1\ldots i_n},$$

and F stands for the number of the Fermi fields involved in $\mathcal{H}(x)$.

• When we reverse the order of multiplication back to the original one, we obtain:

$$\begin{aligned}\mathcal{H}^{CPT}(x) &= e^D \sum_{i_1\ldots i_n} f_{i_1\ldots i_n}\Phi^1_{i_1}(-x_1)\ldots\Phi^n_{i_n}(-x_n)|_{x_1=\ldots=x_n\equiv x} \\ &= \sum_{i_1\ldots i_n} f_{i_1\ldots i_n}\Phi^1_{i_1}(-x)\star\ldots\star\Phi^n_{i_n}(-x) \\ &= \mathcal{H}(-x)\ .\end{aligned}$$

Thus the CPT theorem is valid not only in local field theories but also in noncommutative field theories, for any form of noncommutativity (**general** $\theta^{\mu\nu}$).

- An interacting theory that violates CPT invariance necessarily violates Lorentz invariance.

Greenberg (2002)

- NCQFT - example of theory with Lorentz symmetry violation, but conserved CPT.

Spin-statistics theorem

• Pauli demonstrated the connection between spin and statistics, based on the following requirements:

i) The vacuum is the state of lowest energy;

ii) Physical quantities (observables) commute with each other in two space-time points with a space-like distance ("microcausality");

iii) The metric in the physical Hilbert space is positive definite.

Pauli (1940-1950)

• In NC QFT, the observables which are in general products of several field operators, are no more local quantities and could therefore fail to fulfil the above requirement *ii)*.

• Evaluate equal time commutation relation for an observable such as $:\phi^2(x):$ of a real scalar field of mass m, using Bose statistics:

$$
\begin{aligned}
&\langle 0|[:(\phi\star\phi)(x):\ ,\ :(\phi\star\phi)(y):]\Big|_{x_0=y_0}|p,p'\rangle \\
= &\ -\frac{2i}{(2\pi)^{2d}}\frac{1}{\sqrt{\omega_p\omega_{p'}}}(e^{-ip'x-ipy}+e^{-ipx-ip'y}) \\
\times &\ \int\frac{d\vec{k}}{\omega_k}\sin[\vec{k}(\vec{x}-\vec{y})]\cos\left(\frac{1}{2}\theta_{\mu\nu}k^\mu p^\nu\right)\cos\left(\frac{1}{2}\theta_{\mu\nu}k^\mu p'^\nu\right)
\end{aligned}
$$

• The r.h.s. is nonzero only when $\theta^{0i}\neq 0$. This statement holds for observables consisting of any power of $\phi(x)$ and its derivatives with $\star$-product and also for products of spinor fields $\bar{\psi}(x)$ and $\psi(x)$ (using anti-commutation relation).

• As a result, microcausality (hence possibly spin-statistics theorem) holds for theories with space-space noncommutativity ($\theta_{0i}=0$).

Chaichian, Nishijima, Tureanu (2002)

- *Light-like noncommutativity ($\theta^{\mu\nu}\theta_{\mu\nu} = 0$)*

- case compatible with unitarity
- the integral in the r.h.s. is nonzero

$$\langle 0|[:(\phi\star\phi)(x):\ ,\ :(\phi\star\phi)(y):]\Big|_{x_0=y_0}|p,p'\rangle$$
$$= \frac{\pi}{4}\frac{\cos\left(m\sqrt{(\theta p_2)^2-(\theta p_2-|z_1|)^2-(\theta\omega_p-\theta p_1-|z_2|)^2}\right)}{\sqrt{(\theta p_2)^2-(\theta p_2-|z_1|)^2-(\theta\omega_p-\theta p_1-|z_2|)^2}}\ ,$$

for

$$\begin{aligned} 0 &< |z_1| < \theta p_2\ , \\ \theta(\omega_p-p_1-p_2) &< |z_2| < \theta(\omega_p-p_1)\ , \end{aligned}$$

where $p_\mu = p'_\mu$ ($\mu = 0,1,2$), $x - y \equiv z$ and $\theta \equiv \theta^{02}$.

- If the field theory with light-like noncommutativity is indeed the low-energy limit of string theory, as stated in Aharony, Gomis and Mehen (2000), it is then intriguing that the theory is unitary but acausal (as it is known that a low-energy effective theory should not necessarily be unitary, as is the case, e.g., for the Fermi four-spinor interaction).

Spin in NC QFT?

• Stability group of θ is $SO(1,1) \times SO(2)$

Álvarez-Gaumé, Barbón and Zwicky (2001)

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta' \\ 0 & 0 & -\theta' & 0 \end{pmatrix}$$

• Both $SO(1,1)$ and $SO(2)$ being Abelian groups, they have only one-dimensional unitary irreducible representation and thus no spinor, vector etc. representations.

• Solution: twisted Poincaré symmetry of NC QFT admits the same representation content as usual Poincaré algebra

Chaichian, Kulish, Nishijima and Tureanu (2004)

Chaichian, Prešnajder and Tureanu (2004)

Twisted Poincaré symmetry and spin-statistics relation

- $\mathcal{R}$-matrix relates the coproduct Δ_t and $\Delta_t^{op} = \tau \circ \Delta_t$, τ - flip operator:

$$\mathcal{R}\Delta_t = \Delta_t^{op}\mathcal{R}, \quad \mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 \Rightarrow \mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} = exp(-i\theta^{\mu\nu} P_\mu \otimes P_\nu)$$

- Concept of permutation changes

Chari and Pressley (1994)
Chaichian and Demichev (1996)
Fiore and Schupp (1995)
Kulish and Mudrov (2004)

- Consider now V and W, two (co)representation spaces of the quasi-triangular Hopf algebra $\mathcal{H}$. Then the deformed permutation Ψ is given by

$$\Psi_{V,W}(v \otimes w) = P(\mathcal{R} \triangleright (v \otimes w)),$$

where $\triangleright$ is the action of $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$, followed by P - the usual vector-space permutation.

- Ψ is such that its action commutes with the action of an element of the deformed Hopf algebra $h \in \mathcal{H}$,

$$\begin{aligned} h \bullet \Psi(v \otimes w) : &= \Delta(h) \triangleright P(\mathcal{R} \triangleright (v \otimes w)) = P(\Delta^{op}(h)\mathcal{R} \triangleright (v \otimes w)) \\ &= P(\mathcal{R}\Delta(h) \triangleright (v \otimes w) = \Psi(h \bullet (v \otimes w)). \end{aligned}$$

• In NC QFT with twisted Poincaré symmetry:

$$P \to \Psi(\mathcal{R}) = P\ \mathcal{R} = P\mathcal{F}^{-2}$$

but $\Psi^{-1} = \Psi \Rightarrow$ "symmetric braiding" $\equiv$ no braiding!

• Consider NC free scalar quantum field

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2E_p}} \left[a(\mathbf{p})e^{-ipx} + a^\dagger(\mathbf{p})e^{ipx}\right],$$

and realization of P_μ as quantum momentum operator

$$P_\mu = \int d^3k\, k_\mu\, a^\dagger(\mathbf{k})a(\mathbf{k}), \quad [P_\mu, a(\mathbf{k})] = -k_\mu a(\mathbf{k}), \quad [P_\mu, a^\dagger(\mathbf{k})] = k_\mu a^\dagger(\mathbf{k})$$

- $\star$-product between creation and annihilation operators, but no $\star$-product between exponentials!

$$a^\dagger(\mathbf{k}) \star a^\dagger(\mathbf{p}) = m \circ \mathcal{F}^{-1}\left(a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p})\right) = a^\dagger(\mathbf{k})a^\dagger(\mathbf{p})e^{-\frac{i}{2}k_\mu\theta^{\mu\nu}p_\mu}$$

$$\Rightarrow a^\dagger(k) \star a^\dagger(\mathbf{p}) = a^\dagger(\mathbf{p}) \star a^\dagger(\mathbf{k})e^{-ik_\mu\theta^{\mu\nu}p_\mu}$$

- but $m \circ \mathcal{F}^{-1}\Psi(\mathcal{R})\left(a^\dagger(\mathbf{k}) \otimes a^\dagger(\mathbf{p})\right) = a^\dagger(\mathbf{p}) \star a^\dagger(\mathbf{k})e^{-ik_\mu\theta^{\mu\nu}p_\mu}$

Tureanu (2007)

$\Rightarrow$ statistics OK, shown also directly in

Bu, Kim, Lee, Vac and Yee (2006)

CPT and spin-statistics theorems in axiomatic approach to NC QFT

- Space-space noncommutativity, s.t. only $\theta_{12} = \theta \neq 0$,

x_1, x_2 - NC coordinates, x_0, x_3 - commutative coordinates

- Axioms:

- *local commutativity condition*:

$$\phi_{f_1} \star \phi_{f_2} = \phi_{f_2} \star \phi_{f_1}, \quad \phi_{f_1} \star \phi_{f_2} = \int dx dx' \left(\phi(x) \star \phi(x')\right) f_1(x) f_2(x')$$

where the test functions $f_1(x), f_2(x') \in S^{\beta},\ \beta < 1/2$

Chaichian, Mnatsakanova, Tureanu and Vernov (2007)

Soloviev (2007)

are zero everywhere except on space-like separated finite domains O and O' in the commutative coordinates,

$$\left(x_0 - x_0{}'\right)^2 - \left(x_3 - x_3{}'\right)^2 < 0,$$

but without any restriction in the noncommutative directions x_1 and $x_2 \Rightarrow$ light-wedge locality condition, in accord with perturbative calculations

Chaichian, Nishijima and Tureanu(2002)
Chu, Furuta and Inami (2005)
Greenberg (2005)

- Why not

$$\phi_{f_1} \star \phi_{f_2} = \phi_{f_2} \star \phi_{f_1}, \quad \left(x_0 - x_0{}'\right)^2 - \left(\mathbf{x} - \mathbf{x}'\right)^2 < 0\,?$$

- *spectral condition*

$$Spec(p) = \{(p_0)^2 - (p_3)^2 \geq 0,\ p^0 \geq 0\}$$

- *twisted Poincaré symmetry*: the fields transform under **finite** (global) twisted Poincaré transformations - HOW?

Chaichian, Kulish, Tureanu, Zhang and Zhang (2007)

- parameters $\Lambda^\mu{}_\nu, a^\mu$ of global Poincaré transformations generate the algebra dual to $\mathcal{U}(P)$

$$x^\mu \to \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1$$

- parameters of finite translations do not commute $\Rightarrow$ NONLOCALITY

$$[a^\mu, a^\nu] = i\theta^{\mu\nu} - i\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \theta^{\alpha\beta}$$
$$[\Lambda^\mu{}_\nu, a^\mu] = [\Lambda^\mu{}_\alpha, \Lambda^\nu{}_\beta] = 0$$

Oeckl (2000)

Gonera, Kosinski, Maslanka and Giller (2005)

- use the "substitute" $O(1,1) \times SO(2)$.

• Wightman functions

$$W(x_1, x_2, ..., x_n) = \langle 0|\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle$$

Álvarez-Gaumé and Vázquez-Mozo (2003)

$$W_\star(x_1, x_2, ..., x_n) = \langle 0|\phi(x_1)\star\phi(x_2)\star...\star\phi(x_n)|0\rangle$$

Chaichian, Mnatsakanova, Tureanu and Vernov (2004)

• A QFT can be recovered from its Wightman functions - "reconstruction theorem". Extension to NC case?

• CPT theorem in NC QFT: CPT invariance condition in terms of Wightman functions, e.g. in the case of a neutral scalar field,

$$W_\star(x_1, x_2, ..., x_n) = W_\star(-x_n, ..., -x_2, -x_1) \ ,$$

for any values of x_1, x_2,...,x_n, is equivalent to the weak local commutativity (WLC) condition,

$$W_\star(x_1, x_2, ..., x_n) = W_\star(x_n, ..., x_2, x_1) \ ,$$

where $(x_1 - x_2$, ...,$x_{n-1} - x_n)$ is a Jost point, i.e. x_1, $x_2, ..., x_n$ are mutually space-like separated in the sense of $O(1,1)$.

CPT and spin-statistics theorems in NC QFT proved along the usual lines, using:
- the analytical continuation of Wightman functions to the complex plane only with respect to the commutative x_0 and x_3 coordinates
- space-time inversion is connected to the identity in the complex $O(1,1)$ group
- space inversion in the NC coordinate is a $SO(2)$ transformation.

Thus, one makes heavily use of the similarities between $SO(1,3)$ and $O(1,1)$, which are essential for the proof, and of the fact that the analytical continuation is not affected by the $\star$-product, since the coordinates in which analytical continuation is performed (x_0, x_3) are fully disjoint from the NC plane (x_1, x_2).

Conclusions

CPT and spin-statistics theorems proven in NCQFT

- in Hamiltonian approach

- in axiomatic formulation

- based on twisted Poincaré symmetry considerations