2+1 Quantum Gravity in Chern-Simons formulation

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Based on 2002 old work with E.Buffenoir and K.Noui:
"Hamiltonian Quantization of Chern-Simons theory with $SL(2,\mathbb{C})$ Group"
E.Buffenoir, K.Noui, Ph.Roche, `hep-th/0202121, Class.Quant.Gravity 19 (2002) 4953`

Metric formulation

Let (M, g) a 2+1 Lorentzian manifold, Λ cosmological constant, l_{Pl} the Planck length.

- **Action**

$$S[g, matter] = S_{EH}[g] + \ S_{mat} = \frac{1}{G}\int_M d^3x\ \sqrt{g}(R[g] - 2\Lambda)\ +\ S_{mat}.$$

  This action is invariant under diffeomorphism.

- **Classical Equations of motion**

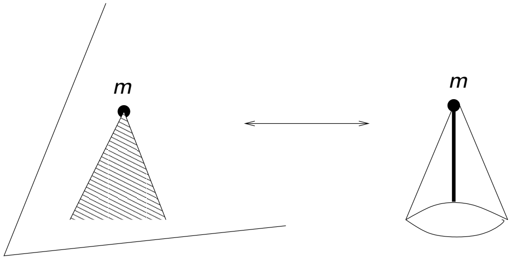
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -GT_{\mu\nu}.$$

  If $\Lambda = 0$ and $T_{\mu\nu} = 0$ the previous equation implies that $R_{\mu\nu} = 0$ which in 2+1 dimension implies that the Riemann tensor is null, i.e the space is locally flat.

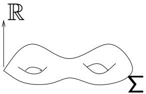
Metric formulation

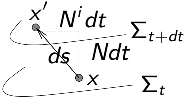
- No local degrees of freedom: no gravitational waves
- Curvature is concentrated at the location of matter: If $\Lambda = 0$ and m is the mass of a point particle, the metric is given by

$$ds^2 = -dt^2 + r^{-8Gm}(dr^2 + r^2 d\theta^2).$$



Hamiltonian formalism





- **A.D.M. decomposition** $M = \Sigma \times \mathbb{R}$

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

- **Canonical momenta**:

  N and N^i appear as Lagrange multipliers

  $\{\pi^{kl}(x); g_{ij}(y)\} = \delta_i^k \delta_j^l \delta(x-y)$

$$S[g] = \int dt \int_\Sigma (\pi^{ij}\dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i)$$

$$\mathcal{H}_i = -2\nabla_j \pi_i^j, \mathcal{H} = \frac{1}{\sqrt{g}} g_{ij} g_{kl}(\pi^{ik}\pi^{jl} - \pi^{ij}\pi^{kl}) - \sqrt{g}(R - 2\lambda).$$

Hamiltonian formalism

$$H[g,\pi;N,N^i]=\int_\Sigma d^2x(N\mathcal{H}+N^i\mathcal{H}_i)$$

with $\mathcal{H}_i\approx 0,\mathcal{H}\approx 0$ first class constraints (the Hamiltonian constraint and the momentum constraints) generating respectively spatial diffeomorphism and time reparametrization.

Very complete review of Quantum Gravity in 2+1 Dimensions in a closed universe is: "Quantum Gravity in 2+1 Dimensions:the case of a Closed Universe" S.Carlip, gr-qc/0409039.

Chern-Simons formulation

One writes the Einstein-Hilbert lagrangian in first order form:

$$S_{EH}[g] = \frac{1}{G}\int (R[\omega]\wedge e - \Lambda e\wedge e\wedge e)$$

with $\omega = \omega_\mu^{IJ} T_{IJ} dx^\mu$ $so(2,1)$ connection and $e^I = e^I_\mu dx^\mu$ orthonormal cotetrad i.e $g = \eta_{IJ} e^I \otimes e^J$.

From the work of E.Witten ("2+1 gravity as an exactly soluble system" (1988)) one define a connection $\mathcal{A}$ on a Lie algebra $\mathfrak{g}_\Lambda = \oplus_K \mathbb{C} P_K \oplus_{IJ} \mathbb{C} T_{IJ}$

$$\mathcal{A}_\mu = e^I_\mu P_I + \omega^{IJ}_\mu T_{IJ}$$

$$T^I = \epsilon^{IJK} T_{JK} \qquad [T^I, T^J] = \epsilon^{IJ}_K T^K, [T^I, P^J] = \epsilon^{IJ}_K P^K, [P^I, P^J] = -\Lambda \epsilon^{IJ}_K T^K.$$

The structure of $\mathfrak{g}_\Lambda$ is

$\mathfrak{g}_{\Lambda>0} = so(3,1)$ isometry group of deSitter space

$\mathfrak{g}_{\Lambda=0} = iso(2,1)$ Poincare group

$\mathfrak{g}_{\Lambda<0} = so(2,2) = so(2,1)\oplus so(2,1)$ isometry group of antideSitter space.

One defines an invariant form Tr on $\mathfrak{g}_\Lambda$ by

$$Tr(T^I T^J) = 0,\ Tr(P^I P^J) = 0,\ Tr(P^I T^J) = \eta^{IJ}.$$

Chern-Simons formulation

$$S_{EH}[e,\omega] = k\int_M Tr(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) = S_{CS}[\mathcal{A}].$$

This equality is the core of the "equivalence" between 2+1 gravity and Chern-Simons theory, $k^{-1} = G\sqrt{\Lambda}$.
Because Chern-Simons theory is a topological quantum field theory heavily studied, one would like to use this equivalence in order to quantize 2+1 gravity.
Difficulties are numerous:

1. The equivalence is not a complete equivalence: there are non isometrical geometries which have associated connections in the same gauge orbit (Matschull).
2. How do we recover sensible physics from CS theory?
3. The Lie algebra $\mathfrak{g}_\Lambda$ is non compact. How do we quantize effectively CS with non compact group?

In order to study the first two problems one has to couple gravity with matter or to add boundaries in order to obtain sensible physics. An interesting result is the computation of Black Hole entropy using CS SO(3,1) theory with SO(3,1) WZW on the boundary by Maldacena, Strominger.

Chern-Simons formulation

The third problem can be solved in the Hamiltonian quantization of CS using non commutative algebras associated to non compact quantum groups.

- Holonomy algebras Nelson-Regge (1989) when Σ is a torus.
- Combinatorial Quantization of Chern-Simons theory (Fock-Rosly, Alekseev-Grosse-Schomerus, Buffenoir-R):Σ is an arbitrary punctured topological surface but the group is compact.
- Combinatorial quantization applied to $\Lambda > 0$ (Buffenoir-Noui-R) and to $\Lambda = 0$ (Bais-Muller-Schroers-Meusburger) for an arbitrary punctured topological surface.

Let $M = \Sigma \times \mathbb{R}$ and G a Lie group (compact or not) of Lie algebra $\mathfrak{g}$. We describe the classical phase space of Chern-Simons theory.
Let $\mathcal{A} = (\mathcal{A}_0, A)$ with A element of the space $\mathbb{A}(\Sigma, G)$ of $\mathfrak{g}$-connection on Σ.

$$S_{CS}[\mathcal{A}] = 2k \int dt \int_\Sigma d^2x \, \epsilon^{ij} Tr(A_i \dot{A}_j + 2\mathcal{A}_0 F_{ij}(A)).$$

$\mathcal{A}_0$ is a Lagrange multiplier and $F_{ij}(A) = \partial_i A_j - \partial_j A_i + [A_i, A_j] \approx 0$ are first class constraints generating Gauge transformations:

$$A^g = gAg^{-1} - dg\, g^{-1}, g \in C^\infty(\Sigma, G).$$

The symplectic structure on $\mathbb{A}(\Sigma, G)$ is given by

$$\{A_i^a(x); A_j^b(y)\} = \frac{1}{k}\epsilon_{ij} t^{ab} \delta(x - y).$$

The classical phase space is the moduli space of flat connections

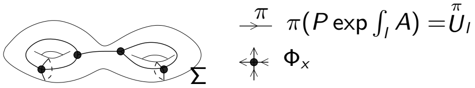
$$M(\Sigma, G) = \{A \in \mathbb{A}(\Sigma, G), F(A) = 0\}/C^\infty(\Sigma, G).$$

Functions on the phase space

Observables: gauge invariant functions of the connection. Wilson loops are examples of observables:

$$\overset{\pi}{W} = tr\left(\pi(P\exp\int_\gamma A)\right)$$

where π is a finite dimensional representation of G.
Spin-Network: a spin-network (flat ribbon graph) is an oriented graph Γ drawn on Σ whose edges l are colored with finite dimensional representations and vertices x are colored by invariant tensor Φ_x of the tensor product of representations incident to the vertex.

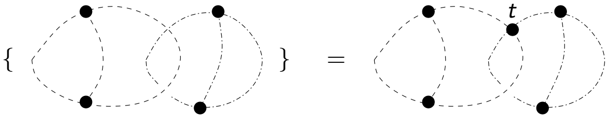


To each spin-network Γ is associated a function on $M(\Sigma, G)$

$$f_\Gamma(A) = \left(\bigotimes_{x \text{ vertex}} \Phi_x\right)\left(\bigotimes_l \overset{\pi_l}{U}_l\right).$$

Goldman Poisson Bracket

This function depends only on the isotopy class of Γ and the Poisson bracket of these functions is given by **Goldman Poisson bracket**:



Problem: Quantize the Poisson algebra $Fun(M(\Sigma, G), \mathbb{C})$ with its structure of Goldman Bracket and find its unitary representations. This can be done by hand in the genus one case and without puncture (*Nelson-Regge*) but needs more elaborate techniques in the general case. This is done using combinatorial quantization.

Principle of the construction: the space $M(\Sigma, G)$ is constructed using a lattice gauge theory on Σ with group G and a quantization $M_q(\Sigma, G)$ is constructed using a deformation of this lattice gauge theory with quantum group "G_q."

1. **discretization of** Σ: let τ a triangulation of Σ ,τ_0 the set of vertices,τ_1 the set of edges, τ_2 the set of faces.
2. **lattice gauge theory**: the space of connections is replaced by the space of discrete connections

$$\mathbb{A}(\tau) = \{U(l) \in G | l \in \tau_1\}$$

3. **Flatness condition**:

$$\forall f \in \tau_2, \prod_{l \in \partial f} U(l) = 1.$$

4. **Gauge group**: let $g \in G^{\tau_0}$, $l = (xy)$

$$U(l)^g = g(x)U(l)g(y)^{-1}$$

$$M(\Sigma, G) = \{U_l \in \mathbb{A}(\tau), \forall f \in \tau_2, \prod_{l \in \partial f} U(l) = 1\} / \prod_{x \in \tau_0} Ad(G_x).$$

The invariant form Tr on $\mathfrak{g}$ defines an invariant tensor $t \in \mathfrak{g}^{\otimes 2}$. We can endow G with a structure of Poisson-Lie group by

$$\{g_1; g_2\} = [r_{12}, g_1 g_2]$$

with $r_{12} + r_{21} = t_{12}$. Fock-Rosly have defined a structure of Poisson on $\mathbb{A}(\tau)$ such that the action of the Poisson-Lie group $\prod_{x \in \tau_0} Ad(G_x)$ is Poisson. It can be shown that the Poisson structure on $M(\Sigma, G)$ that one obtains is independent of the chosen triangulation and is the natural symplectic structure coming from Chern-Simons theory.

Fock-Rosly Poisson structure

$$\{U_1(xz); U_2(xy)\}_{FR} = r_{12} U_1(xz) U_2(xy), l \cap l' = \emptyset \ \ \{U_1(l); U_2(l')\}_{FR} = 0.$$

The definition of $M_q(\Sigma, G)$ is obtained by quantizing Fock-Rosly Poisson structure and it has been defined in a series of work by A.Alekseev, H.Grosse, V.Schomerus. See also the related work on q-Yang Mills by E.Buffenoir and Ph.R. There are two important results:

1. Quantization of $M(\Sigma, G)$ when $\mathfrak{g}$ is a semisimple Lie algebra.
2. Representation theory of $M_q(\Sigma, G)$ when G is a compact group and q is a root of unity.

Very rough idea of the construction. One can define a structure of quantum group $F_q(G)$ which classical limit is given by the Poisson Lie group structure on G, it is given by:

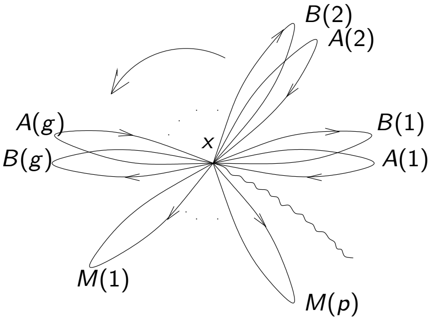
$$\overset{IJ}{R}_{12}\overset{I}{g}_1\overset{J}{g}_2=\overset{J}{g}_2\overset{I}{g}_1\overset{IJ}{R}_{12}, \quad I, J \in Irrep(\mathfrak{g}).$$

One can define a structure of algebra $\mathbb{A}_q(\tau)$ by quantizing Fock-Rosly structure as:

$$\overset{IJ}{R}_{12}\overset{I}{U}_1(xz)\overset{J}{U}_2(xy)=\overset{J}{U}_2(xy)\overset{I}{U}_1(xz),$$

this structure being uniquely defined by imposing that gauge transformation $\mathbb{A}_q(\tau) \rightarrow \mathbb{A}_q(\tau) \otimes \otimes_x F_q(G)_x$ is a morphism of a algebra. By dividing $\mathbb{A}_q(\tau)$ by an ideal imposing the flatness condition and by taking the elements which are invariant under the action of the quantum gauge group one obtains the quantum algebra $M_q(\Sigma, G)$.

In order to built the representation theory of $M_q(\Sigma, G)$ one can take advantage of the fact that the structure of algebra of $M_q(\Sigma, G)$ up to isomorphism does not depend on the triangulation, one can even construct $M_q(\Sigma, G)$ with a graph γ containing only one vertex, one face and $2g$ cycles.



It consists in the holonomies $M(i)$ around the eventual punctures and the holonomies $A(i)$ and $B(i)$ around the handles. The graph algebra is $\mathcal{L}_{p,g} = \mathbb{A}(\gamma)$.

Examples of commutation relations are:

$$R_{12}M_1(i)R_{12}^{-1}M_2(j) = M_2(j)R_{12}M_1(i)R_{12}^{-1}.$$

A fundamental theorem of A.Alekseev is:

$$\mathcal{L}_{p,g} \simeq \mathcal{L}_{1,0}^{\otimes p} \otimes \mathcal{L}_{0,1}^{\otimes g}.$$

Moreover $\mathcal{L}_{1,0} \simeq U_q(\mathfrak{g})$ and $\mathcal{L}_{0,1} \simeq T_q^*(G)$.
As a result an irreducible module of $\mathcal{L}_{p,g}$ is $\otimes_{j=1,\ldots,p} V^{\alpha_j} \otimes F_q(G)^{\otimes g}$ where V^{α_j} are finite dimensional irreducible $U_q(\mathfrak{g})$ modules.

Alekseev-Schomerus

$M_q(\Sigma, G)$ has only one unitary module: $(\otimes_{j=1,\ldots,p} V^{\alpha_j} \otimes F_q(G)^{\otimes g})^{U_q(\mathfrak{g})}$.

This solves completely the problem of constructing a unitary representation of the algebra of observables in Hamiltonian Chern-Simons theory and can be thought of as an explicit implementation of the Dirac program of quantization of a system with first class constraints.
Note that this works perfectly then the group is compact: $\mathfrak{g}$ is endowed with a compact form (a star structure), $M_q(\Sigma, G)$ is a star algebra with q root of unity and the previous representations are finite dimensional unitary representation
How can we modify this construction to the non compact case?

The relation with gravity imposes that $\mathfrak{g}$ is entirely fixed by the choice of sign of Λ. We have chosen $\Lambda > 0$ i.e $\mathfrak{g} = so(3,1)$ but see Bais-Muller-Schroers-Meusburger for the case $\Lambda = 0$ where $\mathfrak{g} = iso(2,1)$. It is easy to see that one must have $q = 1 + \hbar G\sqrt{\Lambda} + o(\hbar)$, as a result we choose $q \in \mathbb{R}^+$. The quantum groups $U_q(so(3,1))$ and $SO(3,1)_q$ have been studied by different people, a very nice definition is to define $U_q(so(3,1))$ as the quantum double of $U_q(su(2)$ following Podles-Woronowicz. Using this quantum double construction we have that $U_q(su(2))$ is a sub-Hopf algebra of $U_q(so(3,1))$. $U_q(so(3,1)_q)$ have two types of irreducible representations which are $U_q(su(2))$ finite:

1. Finite dimensional representations (which are non unitary except the trivial). They are labelled by two spins (I,J) I,J positive half integers.
2. Infinite unitary representations. The principal series and the complementary series. The principal series representation is labelled by $(k, i\rho)$ with k half-integer and $\rho \in \mathbb{R}$.

Our aim is to construct a unitary representation of $M_q(\Sigma, SO(3,1))$.
The definition of $M_q(\Sigma, SO(3,1))$ as a *-algebra is straightforward from the definition of AGS. The elements defining $\mathbb{A}(\tau)$ are $\overset{(I,J)}{U}(l)$ and the R matrix defining the relations are finite dimensional matrices.
The complication arise at the level of the construction of unitary representations.

One can still define a unitary representation of $\mathbb{A}(\tau)$ on the Hilbert space $\mathcal{H} = \otimes_j \overset{\alpha_j}{V} \otimes L^2_q(SO(3,1))^{\otimes g}$ where α_j label principal representation and $L^2(SO(3,1)_q)$ can be defined using the work of Podles-Woronowicz.
However the construction fails because there is no vector in $\mathcal{H}$ which is $U_q(\mathfrak{g})$ because $so(3,1)$ is non compact.
The method that we have designed in Buffenoir-Noui-R to solve this problem is the following:

1. Construct a vector basis of $M_q(\Sigma, SO(3,1))$ using quantum spin networks associated to finite dimensional representations of $U_q(so(3,1))$.
2. Use a Plancherel theorem for the decomposition of $L^2_q(SO(3,1))$ in terms of Principal representations and use the explicit decomposition of the tensor product of principal representations in order to obtain a basis of "Plane waves" belonging to $(\mathcal{H}^*)^{U_q(\mathfrak{g})}$. This basis ψ_χ is labelled by quantum spin networks χ associated to principal representations.
3. Define a representation of $M_q(\Sigma, SO(3,1))$ on wave packets of ψ_χ and find generalized Hermitian scalar product on ψ_χ such that this representation is unitary.

Conclusion

The initial motivation of this work was to study gravity in 2+1 dimension with cosmological constant $\Lambda > 0$. We have proven that one can define rigorously a quantization of Hamiltonian Chern-Simons with $SO(3,1)$ group. There are lots of open questions which could be addressed:

- Classically $d(su(1,1)) = d(su(2)) = so(3,1)$. We have studied of the quantum double of $U_q(su(2)$ but what would happen if we take the quantum double of $U_q(su(1,1))$? Reasons to study this construction.
- Study of the case $\Lambda < 0$. Link with works of Fock,Kashaev,Teschner?
- Analyze more precisely some geometrical construction like grafting in the case where $\lambda \neq 0$ along the line of C.Meusburger.
- One can couple matter to Chern-Simons maintaining integrability. This can be done using the coupling of point particles to Chern-Simons theory along the line of deSousa Gerbert. In the compact case it has been studied by Alekseev-Faddeev-Bytsko in the classical and quantum case. For a recent treatment in the $so(3,1)$ case and in the classical case see (Buffenoir-Noui) and in the quantum case (Buffenoir-R).

All these last works have in common to enlarge the set of observables to *partial* observables and to enlarge the notion of quantum groups to *dynamical* quantum groups.
There has been some recent byproduct of this study, of a purely mathematical interest, concerning the explicit expressions for dynamical cocycle and coboundaries in quantum (affine) Lie algebras (Buffenoir-R-Terras).