Quantum Fields, Curvature, and Cosmology

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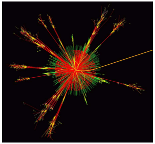
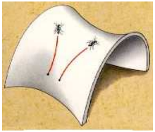
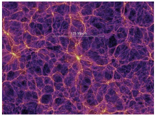
 +  + 

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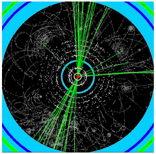
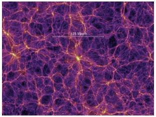
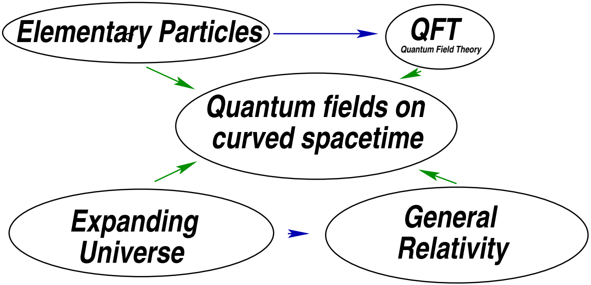
Outline

- Introduction/Motivation
- What is QFT?
- Operator Product Expansions
- Perturbation theory
- Quantum Gauge Theory
- Outlook

Motivation

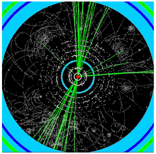


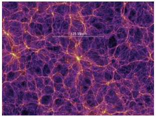


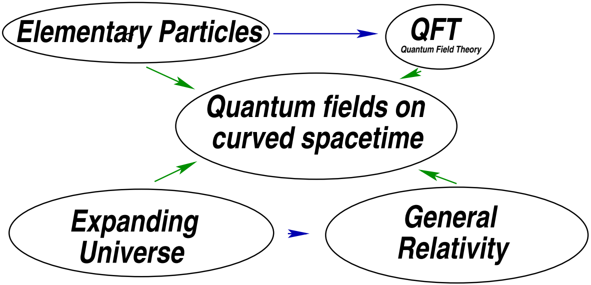


- QFT on manifolds is relevant formalism to describe quantized matter at large spacetime curvature ($\rightarrow$ early Universe).
- Interesting *physical* effects: primordial fluctuations ($\rightarrow$ structure formation, Cosmic Microwave Background, Baryon/Anti-Baryon asymmetry, Hawking/Unruh effect, ...)

Motivation







- QFT on manifolds is relevant formalism to describe quantized matter at large spacetime curvature ($\longrightarrow$ early Universe).
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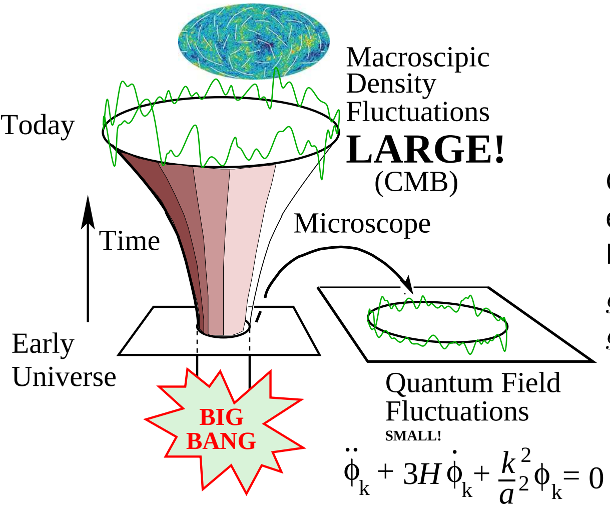
Quantum Fluctuations and Structure of Universe



Consider quantized field eqn. on curved manifold

Example: $\Box_g \phi = 0$

g: Lorentzian metric, e.g. $g = -dt^2 + a(t)^2 ds^2_{\mathbf{R}^3}$.

Why is QFT in curved space so different from flat space?

- No S-matrix
- No natural particle interpretation, no vacuum state
- No spacetime symmetries
- No Hamiltonian/conserved energy (Stability? Thermodynamics?)

$\Longrightarrow$ Forced to a formulation which emphasizes the local, geometrical aspects of QFT.

$\rightarrow$ Algebraic formulation, Operator Product Expansion (OPE) ...: **This talk**

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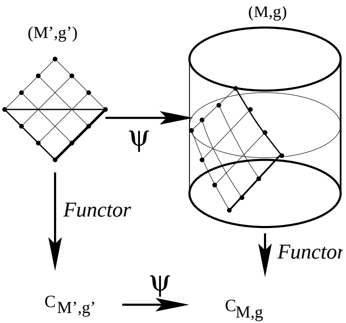
What is QFT?

"Equations" (+ bracket structure)	$\leftrightarrow$	Algebraic relations between quantum fields (OPE)
"Solutions"	$\leftrightarrow$	Quantum states

Example: Free field ϕ:

- OPE: $\phi(x_1)\phi(x_2) \sim H(x_1, x_2)1 + \phi^2(y) + \dots,$ $H = \frac{u}{\sigma + it0} + v \ln(\sigma + it0)$.
- States: Collections of n-point functions $w_n = \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_\Psi$ in which OPE holds.

OPE coefficients have functorial behavior under embedding [S.H. & Wald, Brunetti et al.]:



States do **not** have such a behavior under embedding!

What is the OPE?

General formula: [Wilson, Zimmermann 1969, ..., S.H. 2006]

$$\langle \mathcal{O}_{j_1}(x_1) \cdots \mathcal{O}_{j_n}(x_n) \rangle_\Psi$$
$$\sim \sum \underbrace{C^i_{j_1 \dots j_n}(x_1, \dots, x_n; y)}_{\text{OPE}-\text{coefficients} \leftrightarrow \text{structure``constants''}} \langle \mathcal{O}_i(y) \rangle_\Psi$$

- **Physical idea:** Separate the short distance regime of theory (large "energies") from the energy scale of the state (small) $E^4 \sim \langle \rho \rangle_\Psi$.
- **Application:** In Early Universe have different scales $E \sim T(t) \sim a(t)^{-1}$, curvature radius $R(t) \sim H(t)^{-1}$.
- OPE-coefficients may be calculated within perturbation theory (Yang-Mills-type theories).

Axiomatization of QFT

I propose to **axiomatize** quantum field theory as a collection of operator product coefficients $\{C^{j}_{i_1 \dots i_n}(x_1, \dots, x_n; y)\}$, each of with is the (germ of) a distribution on M^{n+1} subject to

- Covariance
- Local (anti-) commutativity
- Microlocal spectrum condition
- Consistency (Associativity)
- Existence of a state

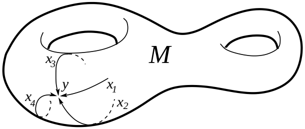
Consequences:

- PCT-theorem holds [S.H. 2003]
- Spin-statistics relation holds [S.H. & Wald 2007]

Short-distance factorization (Consistency)

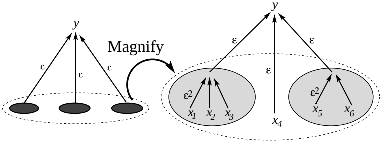
Can scale points in OPE in different ways:

$$\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)$$
$$\sim \sum_i C^i(x_1,\ldots,x_n,y)\mathcal{O}_i(y)$$

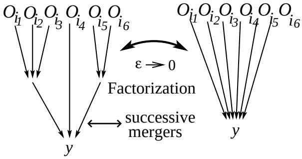


All points scaled towards y

Consider different “merger trees”



Different scalings



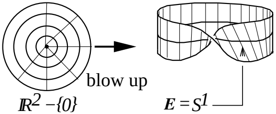
$$C^j_{i_1 i_2 i_3} C^l_{i_4 k j} C^k_{i_5 i_6} = C^l_{i_1 i_2 i_3 i_4 i_5 i_6}$$

Mathematical formulation of associtivity:

"Fulton-MacPherson compactification" [Axelrod & Singer, Fulton & MacPherson]

$\leftrightarrow$ Blow up bndy of configuration space of n points $\mathrm{Conf}[n] = M \times \cdots \times M$ - $\{\mathrm{diagonals}\}$

Example:



For n-point configuration space this leads to

$f_{\mathrm{b.d.}} : M[n] \to \mathrm{Conf}[n]$, with $E[n] = f^{-1}_{\mathrm{b.d.}}(\{\mathrm{diagonals}\})$

$$E[n] = \cup_{\mathrm{trees}} \underbrace{S[\mathrm{merger\,tree}]}_{\text{faces of different dim}} = \mathrm{stratifold}$$

Associativity: OPE-coefficient (pulled back by $f^*_{\mathrm{b.d.}}$) factorizes in particular way on each face of $E[n]$. $\to$ "Operad-like" structure.

Wave front set

OPE-coefficients should satisfy a "μ-local spectrum condition"

[Brunetti et al., SH]

$\leftrightarrow$ positivity of "energy" in tangent space

$\leftrightarrow$ correct "$i\epsilon$-prescription" (domain of holomorphy)

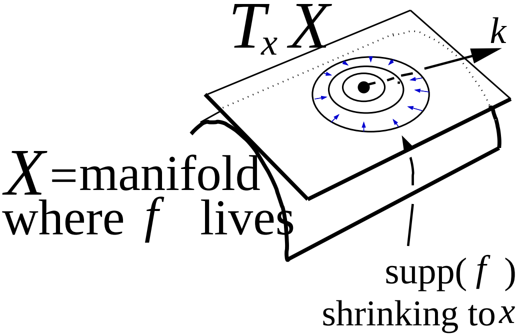
$\leftrightarrow$ (generalized) "Hadamard condition"

Key tool: "Wave front set" [Hö rmander, Duistermaat, Sato, ...]

f smooth, comp. support	$\Longrightarrow$	$\lvert\hat{f}(k)\rvert \sim 1/\lvert k\rvert^N$ all k, all N
f distributional, comp. support	$\Longrightarrow$	$\lvert\hat{f}(k)\rvert \nsim 1/\lvert k\rvert^N$ some k, some N

Wave front set of f at point $x \in X$ defined by

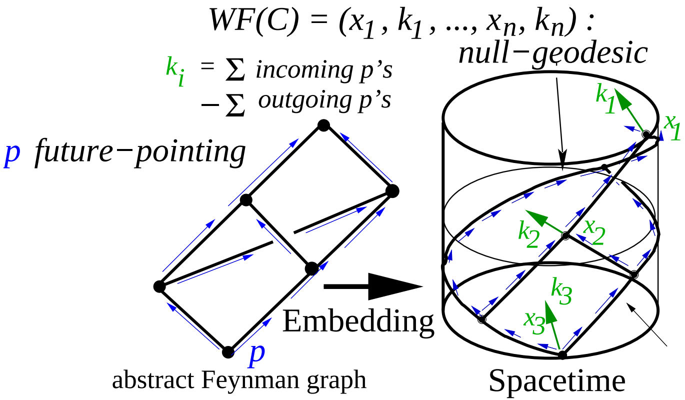
$$\begin{aligned} WF_x(f) &= \{\text{singular directions in momentum space at } x\} \\ &\subset T^*_x X \end{aligned}$$



Wave front set characterizes singularities of f. In QFT typically $X = M^n$ and $f = n$-point function of fields.

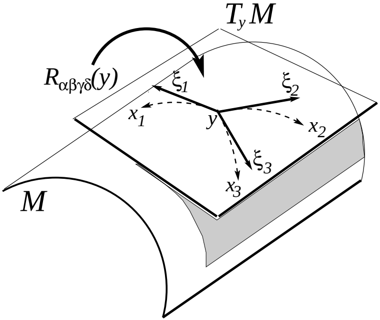
The following μ-**local spectrum condition** [Brunetti et al. 1998,2000] should hold for the OPE coefficients C:

Wave front set $WF(C)$ *has very special form* [S.H. 2006]:



Curvature expansion

$$\begin{aligned}
& C(x_1, \dots, x_n; y) \\
= \quad & \text{structure constants} \\
= \quad & \sum Q[\nabla^k R(y), \text{couplings}] \\
\times \quad & \text{Lorentz inv. Minkowski distributions} \\
& u(\xi_1, \dots, \xi_n)
\end{aligned}$$



ξ_i—Normal coordinates

- Can be computed systematically in pert. theory [Hollands 2006]
- Minkowski distributions $\leftrightarrow$ "Mellin-Moments"

$$u(\xi_1, \dots, \xi_n) = \text{Res}_{z=i\text{power}} \int_0^\infty C(\lambda\xi_1, \dots, \lambda\xi_n, y)\, \lambda^{iz}\, d\lambda$$

Perturbation theory

OPE-coefficients can be constructed in perturbation theory, e.g. scalar field [S.H. 2006]

$$L = d^4x\sqrt{g}[\,|\nabla\phi|^2 + \lambda\phi^4\,]$$

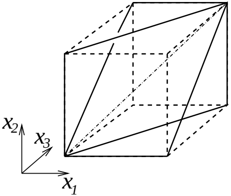
- Given a renormalizable Lagrangian L, can construct OPE coefficients as distributions valued in formal power series.
- Satisfy all above properties.
- Holds in all Hadamard states.
- Also works for Yang-Mills theory [S.H. 2007], but more complicated.

For perturbation theory need **time-ordered products**

$$T_n(\phi^{k_1}(x_1) \otimes \cdots \otimes \phi^{k_n}(x_n)) \in \mathrm{Map}(\mathcal{C}^{\otimes n}, \mathcal{A})$$

Problem: A priori only defined on space

$$M \times \cdots \times M \setminus \bigcup \{\mathrm{diagonals}\}$$



In this viewpoint: extension=renormalization. [Brunetti et al., SH & Wald]

- Combinatorial problem: Diagonals intersect each other $\rightarrow$ "nested divergencies"
- Analytical problem: Must understand singularity structure $\rightarrow$ "wave-front-set," (poly)-logarithmic scaling, ...

Local covariance condition reduces "renormalization ambiguity"

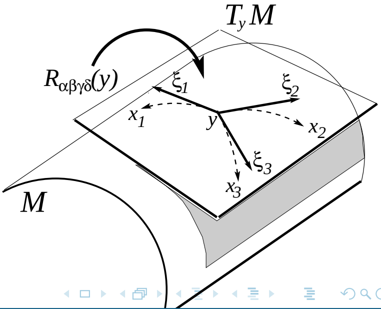
Renormalization

First expansion: time-ordered products

$$T_n(\phi^4(x_1) \otimes \cdots \otimes \phi^4(x_n)) = \sum t_{i_1 \dots i_n}(x_1, \dots, x_n) \underbrace{: \phi^{i_1}(x_1) \cdots \phi^{i_n}(x_n) :}_{\text{cov. def. Wick product}}$$

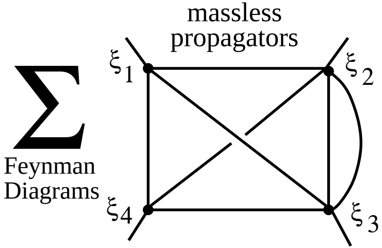
Second expansion: $\mathbb{C}$-valued distributions

$$t(x_1, \dots, x_{n-1}, y) \sim \sum P[\nabla^k R(y), \text{couplings}] \times \text{Lorentz inv. Minkowski distributions } v(\xi_1, \dots, \xi_{n-1})$$

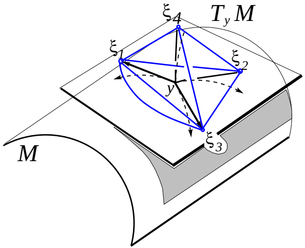


Third expansion: Diagrams

$$v(\xi_1, \dots, \xi_{n-1}) =$$



1. Subdivergences already renormalized.
2. Diagrams “live” in tangent space $T_y M$.
3. E.g. dimensional regularization possible at this stage.



$\Longrightarrow$ Renormalization possible to arbitrary orders!

[S.H. & Wald 2002]

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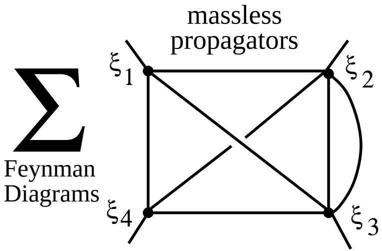

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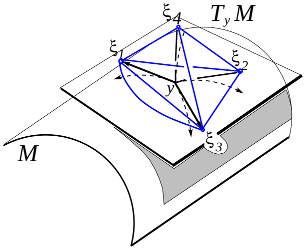

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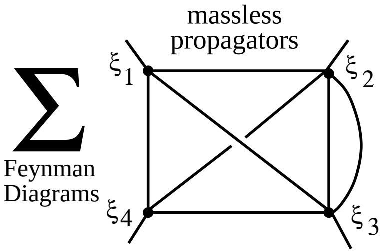

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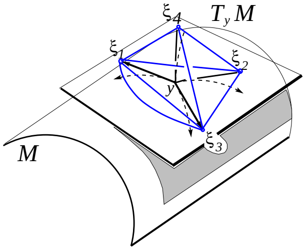

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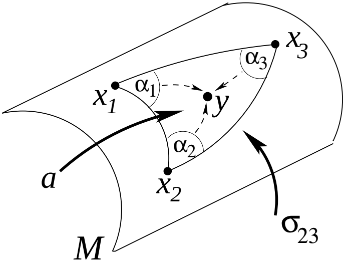
[S.H. & Wald 2002]

Example: 3-point OPE

To leading order in perturbation theory, and leading order in deviation from flat space, 3-point OPE in scalar $\lambda\phi^4$-theory has structure

$$\phi(x_1)\phi(x_2)\phi(x_3) \sim \underbrace{\left[\sum \frac{D}{\sigma_{ij}} + \frac{\lambda}{a} \sum \mathrm{Cl}_2(\alpha_i) + \dots\right]}_{\text{OPE-coefficient } C(x_1,x_2,x_3;y)} \phi(y)$$

(+other operators)



$\mathrm{Cl}_2(z)$—Clausen function

σ_{ij}—geodesic distance

a—curved space area of triangle

D—geometrical determinant

Yang-Mills theory

Can repeat procedure for Yang-Mills theory, $L = d^4x\sqrt{g}\,|F|^2$, with $F = dA + i\lambda[A, A]$ curvature of non-abelian gauge connection.

New issues:

- Need to deal with local gauge invariance $A \to G^{-1}AG + G^{-1}dG$.
- Pass to gauge-fixed theory with additional fields.
- Recover original theory as cohomology of auxiliary theory.
- Need suitable renormalization prescription ($\to$ "Ward identities").

Strategy

- Introduce auxiliary theory $L = L_{ym} + L_{gf} + L_{gh} + L_{af}$, with more fields and BRST-invariance.
- Construct quantized auxiliary theory.
- Define quantum BRST-current J, ensure that $d * J = 0$.
- Define quantum BRST-charge $Q = \int_\Sigma J$, ensure that $Q^2 = 0$.
- Define interacting field observables as cohomology of Q
- OPE closes among gauge invariant operators
- Renormalization group flow (”operator mixing”) closes among gauge-invariant fields.

Ward identities

Construction requires the satisfaction of new set of identities [S.H. 2007]:

$$\left[Q_0, T(e_\otimes^{i\Psi/\hbar})\right] = \frac{1}{2} T\Big((S_0 + \Psi, S_0 + \Psi) \otimes e_\otimes^{i\Psi/\hbar} \Big)$$

where $S = S_0 + \lambda S_1 + \lambda^2 S_2$, and $\Psi = \int f \wedge \mathcal{O}$ is a local observable smeared with cutoff function. Bracket defined by

$$(P, Q) = \int d^4x \sqrt{g} \left(\frac{\delta P}{\delta \phi(x)} \frac{\delta Q}{\delta \phi^\ddagger(x)} \pm (P \leftrightarrow Q) \right)$$

Proof is difficult and requires techniques from relative cohomolgy.

New application of OPE in **curved space**: OPE can e.g. be used in calculations of quantum field theory fluctuations in early universe, where curvature *cannot* be neglected.

Example: Consider $w_3 = \langle \phi\phi\phi \rangle_\Psi$ where ϕ suitable field parametrizing density contrast $\delta\rho/\rho$.

- **Step 1:** Compute OPE-coefficients from perturbation theory (reliable in asymptotically free theories).
- **Step 2:** Write $w_3 \sim \sum C^i \langle \mathcal{O}_i \rangle_\Psi$.
- **Step 3:** Get form factors $\langle \mathcal{O}_i \rangle_\Psi$ e.g. from (a) AdS-CFT, (b) view as input parameters.

Application: Non-Gaussianities in CMB, bispectrum ($\rightarrow$ $f_{NL} = w_3/w_2^{3/2}$ [Shellard,Maldacena,Spergel,...], [Eriksen et al., Bartolo et al., Cabella et al., Gaztanaga et al. (constraints from WMAP data),...]), ...

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Conclusions

- QFT in curved spacetime is a well-developed formalism capable of treating physically interesting interacting models
- Renormalized OPE in curved spacetime available
- Potential applications in Early Universe/cosmology
- Gauge fields can be treated if suitable Ward identities imposed
- Open issues: Supersymmetry, non-pert. regime, singular backgrounds, convergence of pert. series, consistency conditions,...