Wedge-Local Quantum Fields, Non-Commutative Minkowski Space, and Interaction

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"Commutative QFT"	"Non-Commutative QFT"
Spacetime $\mathbb{R}^d$ (Minkowski)	Spacetime $\mathbb{R}^d_\theta$, $[\hat{\boldsymbol{x}}_\mu, \hat{\boldsymbol{x}}_\nu] = i\,\theta_{\mu\nu}$
Quantum Field $\phi(x)$	Quantum Field via Weyl-Moyal: $\phi_\otimes(\theta, x) := \int d^dp\, e^{ip\cdot(\hat{\boldsymbol{x}}+x)} \otimes \tilde{\phi}(p)$
Most important principles: Locality & Poincaré Covariance	Locality & (Lorentz) Covariance are broken
$[\phi(x), \phi(y)] = 0$ if $(x-y)^2 < 0$	$[\phi_\otimes(\theta, x), \phi_\otimes(\theta, y)] \neq 0$ if $(x-y)^2 < 0$
$U(y,\Lambda)\phi(x)U(y,\Lambda)^{-1} = \phi(\Lambda x + y)$	$\Lambda\theta\Lambda^T \neq \theta$ for all Λ

- NC QFT should "look like usual QFT" on larger scales
- Analysis of (remnants of) locality and covariance in NC QFT needed
- Here: Study the situation in a simple model

- Simplest example: Free scalar massive field ϕ in $d = s + 1$ dimensions,

$$\phi_\otimes(\theta, x) = \int \frac{d^s\boldsymbol{p}}{\omega_{\boldsymbol{p}}} \left(e^{ip\cdot x} a^*_\otimes(\theta, p) + e^{-ip\cdot x} a_\otimes(\theta, p) \right) ,$$

$$a_\otimes(\theta, p)^* := e^{ip\cdot\hat{\boldsymbol{x}}} \otimes a(p)^* , \qquad a_\otimes(\theta, p) := e^{-ip\cdot\hat{\boldsymbol{x}}} \otimes a(p)$$

  - $a^\#(p)$: canonical commutation relations
  - $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$, $\boldsymbol{p} \in \mathbb{R}^s$
  - $\mathcal{H}$ =Bose Fock space over one particle space $\mathcal{H}_1 = L^2(\mathbb{R}^s, d^s\boldsymbol{p}/\omega_{\boldsymbol{p}})$
  - Ω =Fock vacuum vector
- Commutation relations of the $a^\#_\otimes(\theta, p)$:

$$a_\otimes(\theta, p) a_\otimes(\theta, p') = e^{-ip\theta p'} a_\otimes(\theta, p') a_\otimes(\theta, p) , \qquad p\theta p' := p_\mu \theta^{\mu\nu} p'_\nu ,$$

$$a_\otimes(\theta, p) a^*_\otimes(\theta, p') = e^{+ip\theta p'} a^*_\otimes(\theta, p') a_\otimes(\theta, p) + \omega_{\boldsymbol{p}} \delta(\boldsymbol{p} - \boldsymbol{p}') \cdot 1 .$$

- $\theta \neq 0 \Longrightarrow \phi_\otimes(\theta, x)$ not local

- n-point functions [Chaichian et.al. 04], [Fiore/Wess 07]

$$\mathscr{W}_n^\theta(x_1, ..., x_n) = \langle \Omega \,, \phi_\otimes(\theta, x_1) \cdots \phi_\otimes(\theta, x_n)\Omega \rangle$$
$$= 1 \cdot \langle \Omega \,, \phi(x_1) \star_\theta \ldots \star_\theta \phi(x_n)\Omega \rangle$$

with $f(x) \star_\theta g(y) = \exp\left(-\frac{i}{2}\partial_x^\mu \theta_{\mu\nu}\partial_y^\nu\right) f(x)g(y)$

"Moyal tensor product"

- amounts to considering a vacuum state of the form $\nu \otimes \langle \Omega \,, \,. \,\Omega \rangle$ on the algebra $\mathcal{F}_\otimes^\theta$ of fields $\phi_\otimes(\theta, x)$
- switch to appropriate (GNS) representation

($\mathcal{F}_\otimes^\theta$ acts reducibly on $\mathcal{V} \otimes \mathcal{H}$)

- Representation on Fock space $\mathcal{H}$, with vacuum vector Ω
- Representation of the fields $\phi_\otimes(\theta, x)$ [Akofor/Balachandran/Jo/Joseph 07, Grosse 79, GL 06, ...]

$$\phi(\theta, x) := \int \frac{d^s\boldsymbol{p}}{\omega_{\boldsymbol{p}}} \left(e^{ip\cdot x}\, a^*(\theta, p) + e^{-ip\cdot x}\, a(\theta, p)\right)$$

$$a(\theta, p) := e^{\frac{i}{2}p\theta P} a(p)\,, \qquad a^*(\theta, p) := e^{-\frac{i}{2}p\theta P} a^*(p)\,,$$

with $(P^\mu\Psi)_n(p_1, ..., p_n) = \sum_{k=1}^n p_k^\mu \cdot \Psi_n(p_1, ..., p_n)\,, \qquad \Psi \in \mathcal{H}$.

- Relation to free field on commutative Minkowski space:

$$\phi(\theta, x) := e^{\frac{1}{2}\partial_x^\mu \theta_{\mu\nu} P^\nu}\, \phi(x)$$

$\phi(\theta, x)$ can be understood as a deformation of $\phi(x)$

- Also possible for general quantum fields ϕ (Buchholz' talk)

- Consider usual "untwisted" representation U of Poincaré group on $\mathcal{H}$: $((y,\Lambda)x = \Lambda x + y$, $j(x) = -x$ total reflection)

$$(U(y,\Lambda)\Psi)_n(p_1,...,p_n) = e^{i\sum_{k=1}^n p_k\cdot y}\Psi_n(\Lambda^{-1}p_1,...,\Lambda^{-1}p_n)$$
$$(U(0,j)\Psi)_n(p_1,...,p_n) = \overline{\Psi_n(p_1,...,p_n)}$$

- Transformation behaviour of $\phi(\theta, x)$ under U:

$$U(y,\Lambda)\phi(\theta,x)U(y,\Lambda)^{-1} = \phi(\gamma_\Lambda(\theta), \Lambda x + y)$$
$$\gamma_\Lambda(\theta) := \begin{cases} \Lambda\theta\Lambda^T & ; \quad \Lambda \in \mathcal{L}^\uparrow \\ -\Lambda\theta\Lambda^T & ; \quad \Lambda \in \mathcal{L}^\downarrow \end{cases}$$

- $\gamma_\Lambda(\theta) = \theta$ for all Lorentz transformations Λ only possible for $\theta = 0$
- $\Longrightarrow \phi(\theta, x)$ is not covariant for $\theta \neq 0$.

Covariance Properties of $\phi(\theta, x)$

- For the "standard θ" in $d = 4$ dimensions,

$$\theta = \theta_1 = \vartheta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \vartheta \neq 0,$$

  we have $\gamma_\Lambda(\theta_1) = \theta_1$ only for
  - $\Lambda =$ Boost in x_1-direction
  - $\Lambda =$ Rotation in x_2-x_3-plane
- Fixed θ breaks Lorentz symmetry $\Rightarrow$ consider family of fields

$$\{\phi(\theta, x) \, : \, \theta \in \Theta\}$$

  with γ-orbit $\Theta = \{\gamma_\Lambda(\theta_1) \, : \, \Lambda \in \mathcal{L}\}$
- get theory with many quantum fields

- Commutation relations between $a^{\#}(\theta, p)$ and $a^{\#}(\theta', p')$ needed

Commutation relations for different θ:

$$a(\theta,p)a(\theta',p') = e^{-\frac{i}{2}p(\theta+\theta')p'}a(\theta',p')a(\theta,p)$$

$$a^*(\theta,p)a^*(\theta',p') = e^{-\frac{i}{2}p(\theta+\theta')p'}a^*(\theta',p')a^*(\theta,p)$$

$$a(\theta,p)a^*(\theta',p') = e^{+\frac{i}{2}p(\theta+\theta')p'}a^*(\theta',p')a(\theta,p) + \omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{p}')e^{\frac{i}{2}p(\theta-\theta')P}$$

- Transformation behaviour $\phi(\theta, x) \to \phi(\gamma_\Lambda(\theta), \Lambda x + y)$ similar to string-localized fields of [Mund/Schroer/Yngvason 05]
- Interpretation of θ as localization region in $\mathbb{R}^d$ possible?

Idea:

Find set $\mathcal{W}_0$ of (causally complete) regions in $\mathbb{R}^d$ and a bijection

$$W : \Theta \to \mathcal{W}_0 \,, \qquad W(\gamma_\Lambda(\theta)) = \Lambda W(\theta) =: \iota_\Lambda(W(\theta))\,.$$

$\longrightarrow$ need isomorphic homogeneous spaces $(\mathcal{W}_0, \iota) \cong (\Theta, \gamma)$

- $W_1 := W(\theta_1)$ must satisfy

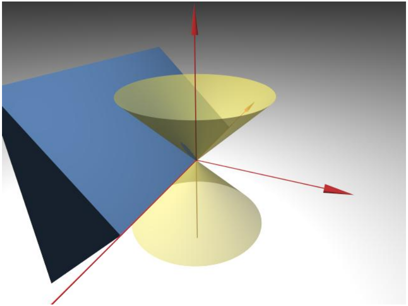
$$\Lambda W_1 = W_1 \quad \text{for} \quad \gamma_\Lambda(\theta_1) = \theta_1$$

  $\longrightarrow$ Condition on the shape of $W_1 \subset \mathbb{R}^4$:
  - $\Lambda W_1 = W_1$ for $\Lambda =$ Boost in x_1-direction
  - $\Lambda W_1 = W_1$ for $\Lambda =$ Rotation in x_2-x_3-plane
- Such a region is well-known: The wedge

$$W_1 = \{x \in \mathbb{R}^d \,:\, x_1 > |x_0|\}$$

Wedges in $\mathbb{R}^d$

- Reference region $W_1 := \{x \in \mathbb{R}^d \,:\, x_1 > |x_0|\}$
- Set of wedges: $\mathcal{W}_0 := \mathcal{L}W_1$ (Lorentz transforms of W_1)
- $W \in \mathcal{W}_0$ satisfies $W' = -W$.
- Pictures in $d = 3$:



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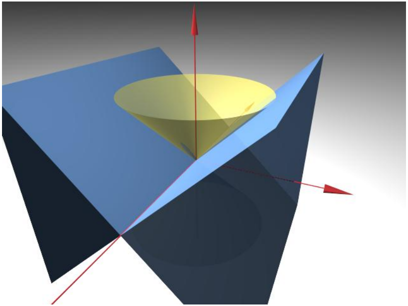

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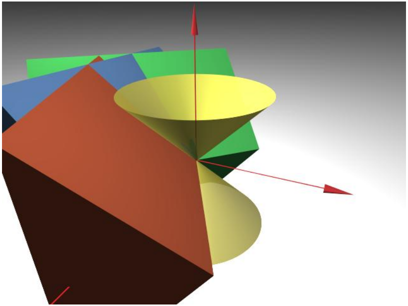

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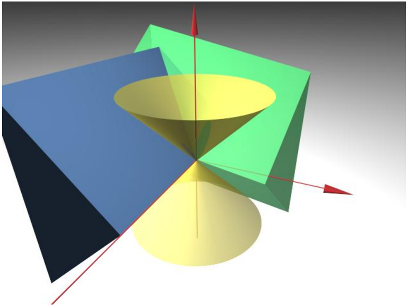

Consider symmetry group $\hat{\mathcal{L}} := \mathcal{L}_+^\uparrow \cup j\mathcal{L}_+^\uparrow$ ($\hat{\mathcal{L}} = \mathcal{L}_+$ in even dimensions)

Proposition

Let $\vartheta \neq 0$. Then

1. $\theta(\Lambda W_1) := \gamma_\Lambda(\theta_1)$ is a well-def. iso. of $\hat{\mathcal{L}}$-homogeneous spaces iff

$$\theta_1 = \vartheta \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} (d \neq 4), \quad \theta_1 = \vartheta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} (d = 4).$$

2. $\theta(W') = -\theta(W), \quad W \in \mathcal{W}_0\,.$

- Matching of symmetries of wedges and nc. params for $d = 2, 3, 4$
- P, T broken in $d = 4$, but TCP not, i.e. $j : x \mapsto -x$ is a symmetry
- Proof of second statement: Recall $W' = -W$.

$$\Longrightarrow \theta(W') = \theta(jW) = \gamma_j(\theta(W)) = -j\theta(W)j^T = -\theta(W).$$

- With the isomorphism $\theta : \mathcal{W}_0 \to \Theta$, define, $W \in \mathcal{W}_0$,

$$\begin{aligned}\phi_W(x) &:= \phi(\theta(W), x) \\ &= \int \frac{d^s\boldsymbol{p}}{\omega_{\boldsymbol{p}}} \Big(a^*(\theta(W), p)\, e^{ip\cdot x} + a(\theta(W), p)\, e^{-ip\cdot x} \Big)\end{aligned}$$

- Direct consequence of construction:

$$\begin{aligned}U(y,\Lambda)\phi_W(x)U(y,\Lambda)^{-1} &= \phi(\gamma_\Lambda(\theta(W)), \Lambda x + y) \\ &= \phi_{\Lambda W}(\Lambda x + y)\end{aligned}$$

- $\phi_W(x)$ describes an extended field configuration in $W + x$
- Is $\phi_W(x)$ localized in $W + x$ in the sense of Einstein, i.e.

$$[\phi_W(x), \phi_{\tilde{W}}(y)] = 0 \quad \text{for} \quad (W + x) \subset (\tilde{W} + y)' \ \ ?$$

- Answer: **Yes!**
- For the proof, use geometrical fact: If $W, \tilde{W} \in \mathcal{W}_0$ are spacelike separated, then

$$W \subset \tilde{W}' \Longrightarrow \tilde{W} = W' + a \quad \text{for some} \quad a \in \mathbb{R}^d$$

$\Longrightarrow$ For proving wedge-locality, it is sufficient to consider

$$[\phi_{W_1}(f), \phi_{W_1'}(g)]\,, \qquad f \in C_0^\infty(W_1),\ g \in C_0^\infty(W_1')$$

- Recall algebra of the $a^\#(\theta, p)$ and $\theta(W') = -\theta(W)$.

$$[a(\theta(W), p),\ a(\theta(W'), p')] = 0$$
$$[a^*(\theta(W), p),\ a^*(\theta(W'), p')] = 0$$
$$[a(\theta(W), p),\ a^*(\theta(W'), p')] = \omega_{\boldsymbol{p}}\delta(\boldsymbol{p} - \boldsymbol{p}')e^{ip\theta(W)P}$$

- Then do analytic continuation from upper to lower mass shell. Result:

Theorem

ϕ_W is a temperate quantum field with the following properties, $W \in \mathcal{W}_0$

1. Covariance: $(y, x \in \mathbb{R}^d,\ \Lambda \in \hat{\mathcal{L}},\ W \in \mathcal{W}_0)$

$$U(y,\Lambda)\phi_W(x)U(y,\Lambda)^{-1} = \phi_{\Lambda W}(\Lambda x + y)\,.$$

2. Wedge-Locality: $(x, y \in \mathbb{R}^d,\ W, \tilde{W} \in \mathcal{W}_0)$

$$(W + x) \subset (\tilde{W} + y)' \Longrightarrow [\phi_W(x), \phi_{\tilde{W}}(y)] = 0\,.$$

3. Reeh-Schlieder property ($\mathcal{O} \subset \mathbb{R}^d$ open, $W \in \mathcal{W}_0$):

$$\overline{\text{span}\{\phi_W(f_1)\cdots\phi_W(f_n)\Omega \ :\ n \in \mathbb{N}_0,\ f_1, ..., f_n \in \mathscr{S}(\mathcal{O})\}} = \mathcal{H}\,.$$

- In $d = 2$, $\mathcal{W}_0 = \{W_1, -W_1\}$ and $\Theta = \{\theta_1, -\theta_1\}$.
  Isomorphism $\theta : \mathcal{W}_0 \longrightarrow \Theta$ is

$$\theta(\pm W_1) = \pm\theta_1 = \pm\vartheta \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} .$$

- Put $p(\beta) := m(\cosh\beta, \sinh\beta) \in H_m^+$. Then

$$e^{ip(\beta_1)\theta_1 p(\beta_2)} = e^{i\vartheta m^2 \sinh(\beta_1 - \beta_2)} =: S_2(\beta_1 - \beta_2)$$

- Let

$$z(\beta) := a(-\theta_1, p(\beta)) , \qquad z(\beta)' := a(\theta_1, p(\beta)) .$$

- Commutation relations:

$$z(\beta_1)z(\beta_2) = S_2(\beta_1-\beta_2)\,z(\beta_2)z(\beta_1)$$
$$z(\beta_1)z^\dagger(\beta_2) = S_2(\beta_2-\beta_1)\,z^\dagger(\beta_2)z(\beta_1) + \delta(\beta_1-\beta_2)\cdot 1$$

$\rightarrow$ representation of Zamolodchikov-Faddeev alg. with sc. fctn S_2

-

$$z(\beta_1)'z(\beta_2)' = S_2(\beta_1-\beta_2)^{-1}\,z(\beta_2)'z(\beta_1)'$$
$$z(\beta_1)'z^\dagger(\beta_2)' = S_2(\beta_2-\beta_1)^{-1}\,z^\dagger(\beta_2)'z(\beta_1)' + \delta(\beta_1-\beta_2)\cdot 1$$

$\rightarrow$ representation of Zamolodchikov-Faddeev alg. with sc. fctn S_2^{-1}

- S_2 is a scattering function for $\vartheta \geq 0$, i.e. bounded & analytic on $\{\zeta\,:\,0 < \mathrm{Im}\,\zeta < \pi\}$ and

$$\overline{S_2(\beta)} = S_2(\beta)^{-1} = S_2(\beta+i\pi) = S_2(-\beta)\,.$$

- In $d = 2$, the algebraic structure coincides precisely with the structure of an integrable QFT with scattering function

$$S_2(\beta) = e^{im^2\vartheta \sinh \beta}$$

[Schroer 97, GL 03, Buchholz/GL 04, GL 06]

- But S_2 is not regular in the sense that it is bounded and analytic on $\{-\kappa < \text{Im}\,\zeta < \pi + \kappa\}$ with $\kappa > 0$
- Known existence/structure theorems for local observables [Buchholz/GL 04, GL 06] do not apply here
- Status of local observables presently unclear

- Non-locality expected from relation to noncommutative spacetime

- Other deformations of the field ϕ are possible, for example

$$a(\theta,p) := \Sigma_\theta(p)a(p)$$

with

$$(\Sigma_\theta(p)\Psi)_n(q_1,...,q_n) := \prod_{k=1}^{n} S_2(\mathrm{Arsinh}(p\theta q_k))^{1/2}\cdot\Psi_n(q_1,...,q_n)\,,$$

and S_2 a (...) scattering function.

- Leads to a slightly different exchange algebra of the form

$$a(\theta,p)a(\theta',p') = A_{\theta\theta'}(p,p')a(\theta',p')a(\theta,p)$$
$$a(\theta,p)a^*(\theta',p') = B_{\theta\theta'}(p,p')a^*(\theta',p')a(\theta,p) + \delta(\boldsymbol{p}-\boldsymbol{p}')\,C_{\theta\theta'}(p)$$

- Covariance, Wedge-Locality and Reeh-Schlieder still holds for

$$\phi(\theta,x) = \int \frac{d^s\boldsymbol{p}}{\omega_{\boldsymbol{p}}}\left(e^{ip\cdot x}\,a^*(\theta,p) + e^{-ip\cdot x}\,a(\theta,p)\right)$$

Interaction in higher dimensions, scattering states

- Given any of these deformed families of quantum fields, what about the interaction? Interpretation of S_2?
- In $d > 1+1$, it is still possible to calculate two-particle scattering
- Method: Haag-Ruelle scattering theory
- Construct two-particle states with the right asymptotic localization and momentum space properties [Borchers/Buchholz/Schroer 00]

- Results: Two-particle scattering states depend on non-commutativity (choice of wedge-fields)

$$(f^+ \times g^+)^W_{\text{out}}(p,q) = e^{-\frac{i}{2}p\,\theta(W)q}\, f^+(p)g^+(q) + e^{\frac{i}{2}p\,\theta(W)q}\, f^+(q)g^+(p)$$

$$(f^+ \times g^+)^W_{\text{in}}(p,q) = e^{\frac{i}{2}p\,\theta(W)q}\, f^+(p)g^+(q) + e^{-\frac{i}{2}p\,\theta(W)q}\, f^+(q)g^+(p)\,.$$

- Similar formulae for the more general deformations (with $a(\theta, p) = \Sigma_\theta(p)a(p)$)
- NC leads to change of S-matrix: non-trivial scattering

- Properties of the deformed S-matrix: Buchholz' talk
- $e^{ip\theta q}$ is phase $\Rightarrow$ No change in cross sections, but in time delays
- Situation similar to integrable models in $d = 1 + 1$
- Wedge-local, covariant, interacting QFT in any dimension

Local observables

- Many wedge-local fields exist, but what about local observables?
- Condition on a (bounded) operator A to be localized in a region $\mathcal{O}$ in Minkowski space:

$$[A, \phi_W(x)] = 0 \quad \text{whenever} \quad \mathcal{O} \subset (W + x)'$$

- Set $\mathcal{A}(\mathcal{O})$ of all solutions of this condition is a (v. Neumann) algebra.
- Local observable content of the model measured by "size" of $\mathcal{A}(\mathcal{O})$ for bounded $\mathcal{O}$

- Reminder: Situation in local QFT:

$$\mathcal{A}(\mathcal{O})\Omega \subset \mathcal{H} \quad \text{is dense (Reeh-Schlieder property)}$$

- Extreme opposite: $\mathcal{A}(\mathcal{O}) = \mathbb{C} \cdot 1$ for all bounded $\mathcal{O}$ (No local observables at all).

Situation in the model at hand:

Proposition

Let $\vartheta \neq 0$.

1. If $d = 1 + 1$, the Reeh-Schlieder property holds, and the theory is completely local.
2. If $d > 1 + 1$, the Reeh-Schlieder property is not valid locally, i.e. $\mathcal{A}(\mathcal{O})\Omega \subset \mathcal{H}$ is not dense if $\mathcal{O}$ is bounded.

- Indirect evidence for non-locality of the model
- Similar situation found in [Buchholz/Summers 06]

- Model defined by the fields ϕ_W is not generated by a local QFT

Conclusions and open questions

New family of model QFTs:

- Related to "free" field on NC Minkowski space
- Consequent application of Poincaré symmetry leads to wedge-local fields
- Remnants of Covariance and Locality found in NC model
- Two-particle S-Matrix becomes non-trivial

Big open question:

- How to do scattering theory on NC spaces in general?
- Notion of "asymptotically commutative" spaces needed / helpful