Faculty of Physics
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ON PARTICLE PHYSICS AND QUANTUM FIELD THEORY

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"COMMUTATIVE AND NONCOMMUTATIVE QUANTUM FIELDS"

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Quantum Spacetime and Noncommutative Geometry

Abstract:

After a survey on the status of Quantum Field Theory on Quantum Spacetime, recent results will be presented on the area and space volume operators, and especially on the Poincare' invariant spacetime volume operator, and on their spectral properties.

The BASIC MODEL OF QUANTUM SPACETIME IS THE SIMPLEST FULL-POINCARÉ COVARIANT MODEL IMPLEMENTING THE

SPACETIME UNCERTAINTY RELATIONS

$$\Uparrow$$

Q. M. + (cl.) G. R.

DFR 1994, 1995

$$[q_\mu, q_\nu] = i Q_{\mu\nu}$$

fulfilling the QUANTUM CONDITIONS:

$\Rightarrow$ THE Δq_μ's should

$\boxed{\text{at least}}$ obey:

$$\Delta q_0 \sum_{j=1}^{3} \Delta q_j \gtrsim 1,$$

$$\sum_{1 \le j < k \le 3} \Delta q_j \cdot \Delta q_k \gtrsim 1$$

$(1 = \lambda_p^2, \quad \lambda_p = \sim 1.6 \times 10^{-33} \text{cm})$

UNCERTAINTY REL $\Longleftarrow$ COMMUTATION RELATIONS:

$$[q_\mu, q_\nu] = i\, Q_{\mu\nu}$$

MORE CAREFUL ARGUMENT INDICATES

$$Q_{\mu\nu} = Q_{\mu\nu}(q)$$

(WHERE THE PRECISE DEPENDENCE IS STILL UNKNOWN), SO THAT

$$[q_\mu, q_\nu] = i\,Q_{\mu\nu}(g)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}(\psi)$$

$$F_g(\psi) = 0$$

ARE COUPLED EQ. OF MOTION!

GEOMETRY ~ DYNAMICS,

ALGEBRA ~ DYNAMICS.

RELATED TO :

- COSMOLOGICAL CONSTANT $\neq 0$;
- ~ EQUILIBRIUM OF CMB WITHOUT INFLATION

S. D. ARXIV 2001, 2006.

$$[q_\mu, Q_{\nu\lambda}] = Q_{\mu\nu} Q^{\mu\nu} = 0,$$

$$\left(\frac{1}{4} Q_{\mu\nu} (*Q)^{\mu\nu}\right)^2 = I$$

$$(\hbar = c = G = 1).$$

$\mathcal{C}_0(\mathbb{R}^4)$ is replaced by $\mathcal{E}$, the ENVELOPPING C^*-ALGEBRA OF THE WEYL RELATIONS

$$e^{i\alpha q} e^{i\beta q} = e^{\frac{i}{2}\alpha Q \beta} e^{i(\alpha+\beta)q},$$

<u>Thm</u> $\mathcal{E} \simeq \mathcal{C}_0(\Sigma, \mathcal{K})$

Σ = REAL ANTISYMM 2-TENSORS

$\sigma_{\mu\nu}$, $\mu, \nu = 0, .., 3$, s.t.

$$e^2 = m^2 \,, \quad (\vec{e} \cdot \vec{m})^2 = 1 \,,$$

$$e_j = \sigma_{0j} \,; \quad m_j = \sigma_{kl} \,;$$

(jkl = 123 or cycl)

$$\Sigma = \Sigma_+ \cup \Sigma_- =$$

FULL LORENTZ ORBIT OF

$$\sigma_0 = \left(\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right) ;$$

corresponding <u>REGULAR</u> imeps:

$$q = \begin{pmatrix} Q \otimes I \\ P \otimes I \\ I \otimes Q \\ I \otimes P \end{pmatrix} ; \quad Q, P =$$

1-dim. SCHROEDINGER OPS.

IN THIS REP. :

$$\sum_{\mu=0}^{3} q_\mu^2 =$$

$$= 2(H \otimes I + I \otimes H),$$

$$H \equiv \frac{1}{2}(p^2 + Q^2),$$

$$\sigma(H) = \frac{1}{2} + \mathbb{N}_0 ; \Rightarrow$$

$$\sum_{\mu=0}^{3} q_\mu^2 \geq 2$$

WHICH HOLDS IN ANY REP.!

"=" PRECISELY IN REPS CARRIED BY Σ_1, THE BASE

$$\underbrace{\left\{ \sigma \in \Sigma,\ \sigma = (\vec{e}, \vec{m}),\ e^2 = m^2 = 1 \right\}}_{\shortparallel} \text{ OF } \Sigma,$$

= FULL ROTATION ORBIT OF σ_0.

$$\Sigma \sim T\Sigma_1 \sim TS_2 \times \{\pm 1\}.$$

INDEP. EVENTS :

$$\mathcal{E} \otimes_{\mathcal{Z}} \mathcal{E} \otimes_{\mathcal{Z}} \mathcal{E} \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} \mathcal{E}$$

$\mathcal{Z}$ = CENTRE $M(\mathcal{E}) \sim$

$\sim \mathcal{C}_B(\Sigma)$; i.e.

$$q_j^\mu = I \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} q^\mu \otimes_{\mathcal{Z}} \cdots \otimes_{\mathcal{Z}} I,$$

j-th place

so that

$$[q_j^\mu, q_k^\nu] = i\,\delta_{jk}\,Q^{\mu\nu}$$

INDEP. OF j.

<u>THEN</u>

$$\sum_{\mu=0}^{3} (q_j - q_k)_\mu^2 \geq 4$$

Calculus:

$$f(q) \equiv \int \check{f}(\alpha) e^{i\alpha q} d^4\alpha;$$

$$\int d^4 q \quad = Tr$$

$$\int d^3 q \quad = \lim \int d^4 q \, f_n^{*} \cdot f_n :$$

$$q_0 = t \quad \int g(\alpha) e^{i\alpha q} d^4\alpha \longrightarrow \int e^{-i\alpha_0 t} g(\alpha_0, \vec{0}) \, d\alpha_0$$

QFT:

$$\phi(q) \equiv \int \check{\phi}(k) e^{ikq} d^4 q$$

FREE FIELDS - NON LOCAL BUT POINCARÉ COV.
- LOCAL ALGEBRAS, ...

INTERACTION ?

MUST DEFINE INT. DENSITY

$\mathcal{H}_I(q)$, then

"$H_I(t) = \int_{q_0=t} d^3q \; \mathcal{H}_I(q)$;"

GELL-MANN LOW ANSATZ:

$$S = T \exp i \int_{-\infty}^{\infty} H_I(t)\, dt.$$

DYSON EXPANSION!

!

DFR '94-'95: ϕ^n interactions,

$\mathcal{H}_I(q) = g :\phi(q)^n:$

STILL REQUIRES $\int_\Sigma$!

NO LORENTZ INV. INTEGRATION!

ANSATZ:

$$H_I(t) = \int_{\Sigma_1} d\sigma \int_{q_0=t} d^3q \; \mathcal{H}_I(q)$$

MILD UV REG. (DFR), BUT
SUFFICIENT FOR ϕ^3 (D. BAHNS).
2003

BUT: $:\phi^n(x): =$

$$= \lim_{x_j - x_k \to 0} :\phi(x_1)\ldots\phi(x_n):$$

ON QST: $\sum (q_j - q_k)_\mu^2 \geq 4!$

ALTERNATIVE:

$$H_I(q) \equiv g\, E^{(n)} :\phi(q_1)\ldots\phi(q_n):$$
$$\equiv g :\phi^n(q):_Q$$

E^n = QUANTUM DIAGONAL MAP;

$$\underbrace{\mathcal{E} \otimes_Z \ldots \otimes_Z \mathcal{E}}_{n \text{ times}} \longrightarrow \mathcal{E} \text{ by:}$$

first $\frac{1}{\sqrt{n}} \sum q_j \longrightarrow q \otimes_Z I \otimes_Z \ldots \otimes_Z I$;
($(n+1)$ fold $\otimes$)

~~then~~

$$q_j - q_k \longrightarrow I_Z \otimes (q_j - q_k)$$

then evaluate $id \otimes \underbrace{\eta \otimes \ldots \otimes \eta}_{n\text{-times}}$

where

$$\eta : \int f(Q,\alpha)\, e^{i\alpha q}\, d^4\alpha \longrightarrow$$

$$\longrightarrow \int f(Q,\alpha)\, e^{-\frac{1}{4}|\alpha|^2}\, d^4\alpha$$

UNIVERSAL MAP S.T., IN REP. GNS OF ω

$$\sum q_\mu^2 = 2 \quad (= \text{MIN.})$$

IFF $$\omega = \int_{\Sigma_1} d\mu \circ \eta$$

for some prob. measure on Σ_1.

THUS THE Q. DIAG. MAP

BRINGS q_j close to q_k AS

MUCH AS ALLOWED BY ST UR.

THE THEORY WITH

$$H_I(q) = g :\phi^n(q):_Q$$

IS UV - FINITE.

WITH ADIABATIC DAMPING

THAT IS, THE PERT. EXP. OF

$$\langle \Omega, \int_{-\infty}^{\infty} g(t) H_I(t)\, dt\, \Omega \rangle$$

VACUUM → VACUUM

IS TERM BY TERM UV-FINITE.

(D. BAHNS, K. FREDENHAGEN, G. PIACITELLI, S.D. 2003)

HAPPY? NOT YET:

- BADLY BREAKS LORENTZ INV.
- AD. LIMIT $g \to 1$ IS A PROBLEM (BUT: KOSSOW).

OTHER APPROACHES:

- YANG FELDMAN (BDFP, 2002)
- QUASIPLANAR WICK PRODUCTS (BDFP, 2005 + IN PREP.)

ALL MEET (SOONER OR LATER) PROBLEMS WITH LORENTZ INVARIANCE.

IS QST TO BE ABANDONED? NO!

- NO LESS PROBLEMS THAN OTHER APP.

- IMPOSED BY FIRST PRINCIPLES

- LEADS TO SURPRISINGLY NICE BASIC GEOMETRY:

IN PARTICULAR THE LORENTZ-INVARIANT 4-SPACETIME VOLUME OPERATOR

$$dq \wedge dq \wedge dq \wedge dq$$

HAS A PURE DISCRETE SPECTRUM WITH A GAP OF ORDER 1 (i.e. λ_P^4) FROM 0.

(S.D, K. FREDENHAGEN in prep)

- $\sum_{\mu=0}^{3} |dq^\mu|^2 \geq 4$;

- $\sum_{1\leq j<k\leq 3} |dq_j \wedge dq_k|^2 \geq 1$,
  $\sum_{j=1}^{3} |dq_0 \wedge dq_j|^2 \geq 1$;

- $\sigma(|dq^1 \wedge dq^2 \wedge dq^3|) = [0, +\infty)$;

- $dq \wedge dq \wedge dq \wedge dq$ IS A NORMAL OPERATOR WITH PURE POINT SPECTRUM;

$$\sigma_p(dq \wedge \ldots \wedge dq) = \pm 2 + \mathbb{Z}ab + i(\mathbb{Z}a + \mathbb{Z}b)$$

$$= \pm 2 + \mathbb{Z}\sqrt{5} + i\left(\mathbb{Z}\sqrt{5-2\sqrt{5}} + \mathbb{Z}\sqrt{5+2\sqrt{5}}\right);$$

so that

$$|dq \wedge \ldots \wedge dq| \geq \sqrt{5} - 2.$$

$a^2 + b^2 = (ab)^2 = 5$

SPECTRUM WHERE ?

2. NC DIFF CALCULUS

$\mathcal{Q}$ unital (C^*-) algebra

Def

$$\Lambda(\mathcal{Q}) = \bigoplus_{n=0}^{\infty} \underbrace{\mathcal{Q} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{Q}}_{n\text{-times}}$$

(completed with the min product C^*-norm) each summand is a (C^*-) algebra in its own right.

It is also an $\mathcal{Q}$ - bimodule

$$a \cdot (a_1 \otimes \cdots \otimes a_n) = a a_1 \otimes \cdots \otimes a_n$$

$$(a_1 \otimes \cdots \otimes a_n) \cdot b = a_1 \otimes \cdots \otimes a_n b ,$$

and so is $\Lambda(\mathcal{Q})$. MOTIVATION!

We equip $\Lambda(\mathcal{Q})$ with the $\mathcal{Q}$ - bimodule tensor product :

$$(a_1 \otimes \ldots \otimes a_n)(b_1 \otimes \ldots \otimes b_m) \equiv$$

$$a_1 \otimes \ldots \otimes a_n b_1 \otimes b_2 \otimes \ldots \otimes b_m,$$

$$\Lambda_n(\mathcal{U}) \times \Lambda_m(\mathcal{U}) \to \Lambda_{n+m-1}(\mathcal{U}).$$

Consider

$$da \equiv I \times a - a \otimes I,$$

$$a \in \mathcal{U}$$

as element of $\Lambda(\mathcal{U})$.

Def $\Omega(\mathcal{U}) \equiv$ d-stable subalg gen by $\mathcal{U}$

can be shown $\equiv \bigcap \ker m_k$,

where m_k is def on Λ_m, $m \geq k$, by:

$$\underbrace{m_k(a_0 \otimes \ldots \otimes a_n)}_{\in \Lambda_{n+1}} = \underbrace{a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_n}_{\in \Lambda_n}.$$

Want to define pairing

$$\Lambda(\mathcal{O}) \times \Lambda(\mathcal{O}) \longrightarrow \mathcal{O}$$

which gives "scalar product" on $\Omega(\mathcal{O})$.

"SHUFFLING PRODUCT":

$$\langle a_1 \otimes \cdots \otimes a_m , b_1 \otimes \cdots \otimes b_n \rangle$$

$$= \delta_{nm} \prod_{i=1}^{n} a_i b_i .$$

Remark: let $m = n = 1$,

$$\Lambda_1(\mathcal{O}) = \mathcal{O}, \quad \langle a, b \rangle = ab.$$

let $m = n = 2$,

$$da = I \otimes a - a \otimes I \in \Lambda_2(\mathcal{O}),$$

$$db = I \otimes b - b \otimes I \in \Lambda_2(\mathcal{O}),$$

$$\langle da, db \rangle = ab - ba - ab + ab$$

$$= [a, b]$$

More generally,

$$\langle a\, da_1 \dots da_n \,,\, db_1 \dots db_n \cdot b \rangle =$$

$$= \quad a \prod_{j=1}^{n} [a_j, b_j]\; b \;;$$

other expressions like

$$\langle da_1 \dots da_n \cdot a \,,\, b \cdot db_1 \dots db_n \rangle$$

are easily computed by repeated use of the Leibnitz rule

$$d(ab) = a\, db + (da)\cdot b$$

Classically: combine pairing with INTEGRATION to get $(\cdot\,,\,\cdot)$ and the associated TRANSPOSE δ of d:

δ : n forms $\to$ $n-1$ forms ;

$d\delta + \delta d = (d+\delta)^2 =$ LAPLACIAN

$$\boxed{\text{INTEGRATION} \longrightarrow \text{TRACE on } \Omega}$$

$\mathrm{Tr} : \mathcal{O} \to \mathbb{C}$,

$\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$

(no positivity req.).

Remark : NOT necessary it is $\mathbb{C}$-valued. The universal trace

$$TR : \mathcal{O} \to \mathcal{O} / \{\text{LIN SPAN } [a,b],\ a,b \in \mathcal{O}\}$$

would do.

More concretely if Z = Centre $\mathcal{O}$ ($Z \subset$ Centre of the MULTIPLIER ALGEBRA OF $\mathcal{O}$ if $\mathcal{O}$ had no unit) we can have Z-valued TRACE :

$\mathrm{Tr} : \mathcal{O} \to Z$

Also we can work with

$\Omega \otimes_{\mathbb{Z}} \Omega =$

$\Omega \otimes_{\mathbb{C}} \Omega$ mod two sided ideal

gen by $z \otimes I - I \otimes z$, $z \in \mathcal{Z}$

Yields to $\Lambda_{\mathcal{Z}}(\Omega) \supset \Omega_{\mathcal{Z}}(\Omega)$

where (characterization)

$$dz = 0 \quad , \quad z \in \mathcal{Z}.$$

Clearly the schuffling prod.

$$\Lambda_{\mathcal{Z}} \times \Lambda_{\mathcal{Z}} \longrightarrow \Omega$$

is multilinear on $\mathcal{Z}$.

CODIFFERENTIAL associated to the shuffling product:

$$Tr\langle \delta\omega, \varphi \rangle = Tr\langle \omega, d\varphi \rangle$$

PROP. The HOCHSCHILD BOUNDARY:

$$\delta(a_1 \otimes \dots \otimes a_n) =$$

$$= \sum (-1)^k a_1 \otimes \dots \otimes a_k a_{k+1} \otimes \dots \otimes a_n$$

$$+ (-1)^n a_n a_1 \otimes a_2 \dots \otimes a_n$$

is a codifferential.

Good news end here.

Bad news:

the "quantum wave equation"

$$(d\delta + \delta d)\,\varphi = 0$$

does not seem of use for physics.

BUT THERE IS A LOT OF GOOD USE FOR $\Lambda(\Omega)$ AND SCHAFFLING PRODUCT.

NOTE $\bigwedge_{\mathcal{Z}}(\mathcal{E})$ has

two alg. structures:

- that of $\mathcal{Z}$-bimod. tensor alg
- that of $\bigoplus_n \mathcal{E}^{\otimes_{\mathcal{Z}} n}$;

$dq \wedge .. \wedge dq$ has to be computed with the first prod; it produces elements whose norms spectra, .. are evaluated in the (C*-completion of the) second.

Area : $j = 1, 2, 3$;

$$dq^j \wedge dq^k = (I \otimes q^j - q^j \otimes I)(I \otimes q^k - q^k \otimes I) - (j \rightleftarrows k) =$$

$$= I \otimes q^j \otimes q^k - I \otimes q^j q^k \otimes I - q^j \otimes I \otimes q^k + q^j \otimes q^k \otimes I - (j \rightleftarrows k) =$$

$$\text{s.a.} \quad -i\, Q^{jk}$$

Hence, as an operator in

$$\mathcal{E} \otimes_{\mathcal{Z}} \mathcal{E} \otimes_{\mathcal{Z}} \mathcal{E},$$

$$|dq^j \wedge dq^k|^2 = (\text{s.a.})^2 + Q^{jk\,2}$$

$$\geq Q^{jk\,2},$$

and

$$\sum_{\substack{j,k,l \text{ cyclic} \\ \text{perm of } 1,2,3}} |dq^j \wedge dq^k|^2 \geq \vec{m}^2 \geq I;$$

Similarly for the "timelike area"

$$\sum_{j=1}^{3} |dq^0 \wedge dq^j|^2 \geq \vec{e}^2 \geq I.$$

Space Volume:

$$d\vec{q} \wedge d\vec{q} \wedge d\vec{q} = \varepsilon_{jkl}\, dq^j dq^k dq^l =$$

$$= \varepsilon_{jkl} (I \otimes q^j_{\mathcal{E}} - q^j \otimes I)(I \otimes q^k - q^k \otimes I)(I \otimes q^l - q^l \otimes I)$$

$=$ Σ terms with q's at 3 different tensor places (s.a.) +

+ Σ terms with $q_k q_\ell$ at one place and q_j at another (skew adjoint by antisymmetrization).

$$\equiv \mathcal{V} + i\, m_\ell \hat{Q}^\ell,$$

$$m_\ell = \frac{1}{2} \varepsilon_{\ell jk} Q^{jk},$$

$$\hat{Q}_\ell = \frac{1}{2} \sum_{j=1}^{4} (-1)^{j+1} q_j^\ell,$$

with $q_j = I \otimes .. \underset{j\text{th place}}{q} \otimes .. I$.

CLAIM: $\mathcal{O} \in Sp\,(d\vec{q} \wedge d\vec{q} \wedge d\vec{q})$.

Suffices to work with irreps $\sigma \in \Sigma$.

Choose $\sigma = (\vec{e}, \vec{m})$, with

$\vec{m} = \vec{e} = (1,0,0)$;

The associated regular irrep acts on

$H \otimes H$, $H \equiv L^2(\mathbb{R}, ds)$ as

$$\begin{pmatrix} q^0 \\ \vdots \\ q^3 \end{pmatrix} = \begin{pmatrix} q \otimes I \\ p \otimes I \\ I \otimes q \\ I \otimes p \end{pmatrix},$$

q = multipl by s,

$p = -i\frac{d}{ds}$.

We identify, by the obvious $\cong$,

$(H \otimes H)^{\otimes 4}$ with $H^{\otimes 4} \otimes H^{\otimes 4}$;

Then, in the chosen irrep,

$$m_e \hat{Q}_e = m_1 \hat{Q}_1 = \frac{1}{2} \sum (-1)^{j+1} p_j \otimes I_j,$$

$$V = \sum_{m=1}^{4} (-1)^{m+1} \varepsilon_{jkl}\, q_1^j \cdots q_4^l$$

~~mmmmm~~ I at m-th place

$$= \det \begin{pmatrix} I & p_1 \otimes I & I \otimes q_1 & I \otimes p_1 \\ I & p_2 \otimes I & I \otimes q_2 & I \otimes p_2 \\ I & p_3 \otimes I & I \otimes q_3 & I \otimes p_3 \\ I & p_4 \otimes I & I \otimes q_4 & I \otimes p_4 \end{pmatrix}$$

$$= \sum \varepsilon^{jk\ell m}\, p_k \otimes q_\ell p_m =$$

$$= \frac{1}{4} \sum \varepsilon^{jk\ell m} (p_k - p_j) \otimes M_{\ell m},$$

so that

$$d\vec{q} \wedge d\vec{q} \wedge d\vec{q} = \frac{1}{2} \sum (-1)^{j+1} p_j \otimes I +$$

$$+ \frac{1}{4} \sum \varepsilon^{jk\ell m} (p_k - p_j) \otimes M_{\ell m}.$$

Choose $x, y \in H^{\otimes 4}$, $\|x\| = \|y\| = 1$, s.t. $y \in H^{\otimes 4} = L^2(\mathbb{R}^4, d^4 s)$ depends upon $\sum_{j=1}^{4} s_j^2$ only, and $\hat{x}$ has supp in a sphere of radius $\varepsilon/2$ around 0; then

$$\| d\vec{q} \wedge d\vec{q} \wedge d\vec{q}\; x \otimes y \| \leq \varepsilon .$$

Note: on <u>any</u> y in the domain of M_{em}, $e, m = 1, \dots, 4$, we have, if x is as before,

$$\| d\vec{q} \wedge d\vec{q} \wedge d\vec{q} \; x \otimes y \| \leq$$

$$\leq \quad \varepsilon \left(1 + \frac{1}{3} \sup \| M_{em} y \| \right).$$

The Spacetime Volume Operator

$$dq \wedge dq \wedge dq \wedge dq =$$

$$= \varepsilon_{\mu\nu\lambda\rho}\, dq^\mu dq^\nu dq^\lambda dq^\rho =$$

$$= \varepsilon_{\mu\nu\lambda\rho} \left(I \otimes q^\mu - q^\mu \otimes I \right) \dots \left(I \otimes q^\rho - q^\rho \otimes I \right)$$

(products in $\Lambda_{\mathcal{Z}}(\mathcal{E})$!).

$=$ (no pair of q's in same place) $+$ S. a.

A $\equiv$ (two pairs of q's in same places) $+$ S. a.

(one pair of q's in same place) skew adjoint

$\equiv$ B

$$\varepsilon_{\mu\nu\lambda\rho}\, I \otimes q^\mu q^\nu \otimes I \otimes q^\lambda q^\rho \otimes I =$$

$$= \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho}\, I \otimes [q^\mu, q^\nu] \otimes I \otimes [q^\lambda, q^\rho] \otimes I =$$

$$= -\frac{1}{4}\, Q \wedge Q = -2\eta, \ \underline{\in \mathcal{Z}},$$

$$\eta = \pm I \text{ on } \Sigma_\pm .$$

$$dq \wedge dq \wedge dq \wedge dq = A - 2\eta + iB,$$

$$A = \sum_{j=1}^{5} (-1)^j \bigwedge_{i \neq j} q_i \equiv \sum_{j=1}^{5} (-1)^j A_j,$$

$$B = \frac{1}{2} \sum_{i<j} (-1)^{i-j}\, Q \wedge q_i \wedge q_j \equiv \frac{1}{2}\sum (-1)^{i-j} B_{ij}.$$

Compute:

$$[q_k^\mu, B_{ij}] = \delta_{ik} \underbrace{Q^{\mu\cdot} \wedge Q \wedge q_j}_{} - \delta_{jk}\, Q^{\mu\cdot} \wedge Q \wedge q_i$$

$$\varepsilon_{\nu\lambda\rho\sigma}\, Q^{\mu\nu} Q^{\lambda\rho} q_j^\sigma =$$

$$= \quad 2\left(Q^{\mu\nu}(*Q)_{\nu\sigma}\right) q_j^\sigma$$

But antisymmetry + centrality of $Q \Rightarrow$

$$Q^{\mu\nu}(*Q)_{\nu\sigma} = \frac{1}{4}\, Q^{\nu\rho}(*Q)_{\nu\rho} \cdot \delta_{\mu\sigma}$$

$$= \eta\, \delta_{\mu\sigma}; \quad \text{hence}$$

$$[q_k^\mu, B_{ij}] = \eta\left(\delta_{ik} q_j^\mu - \delta_{jk} q_i^\mu\right). \quad *$$

Thus $\mathrm{Ad}\, B_{ij}$ act on q_k's as ($2\cdot$) Lie Algebra gen of $SO(5)$; by $*$,

$$(\mathrm{Ad}\, B)\left(\sum_{j=1}^{5} q_j\right) = 0$$

hence $\mathrm{Ad}\, B$ acts as a generator of 1 par sbgp in $N \equiv$ stabilizer in $SO(5)$ of $(1,1,1,1,1)$.

Now $A = \sum_{j=1}^{5} (-1)^j \bigwedge_{i \neq j} q_i =$

$$= \det \begin{pmatrix} I & q_1^0 & q_1^1 & \cdots & q_1^3 \\ I & q_2^0 & q_2^1 & \cdots & q_2^3 \\ - & - & - & - & - \\ I & q_5^0 & q_5^1 & \cdots & q_5^3 \end{pmatrix} =$$

$$= \det R \cdot \left(\quad \right)$$

and, if $R \in N$:

$$= \det \begin{pmatrix} I & q'^{0}_{1} & q'^{1}_{1} & \cdots & q'^{3}_{1} \\ I & q'^{0}_{2} & q'^{1}_{2} & \cdots & q'^{3}_{2} \\ - & - & - & - & - \\ I & q'^{0}_{5} & q'^{1}_{5} & \cdots & q'^{3}_{5} \end{pmatrix},$$

if $$q'^{\mu}_{j} = R^{jk} q^{\mu}_{k},$$

$$R \in N,$$

and for any generator D of a 1-par sgp in N,

$$(A \circ D)(A) = 0.$$

Hence $[A, B] = 0$ and

$dq \wedge dq \wedge dq \wedge dq = A - 2z + iB$
is NORMAL (z is central!), and

$$|dq \wedge dq \wedge dq \wedge dq|^2 = (A-2z)^2 + B^2$$
$$\geq (A-2z)^2 .$$

Now as a field of operators on Σ, by LORENTZ invariance, $dq \wedge dq \wedge dq \wedge dq$ is CONSTANT and of opposite signs on $\Sigma_\pm$; it suffices to compute at $\sigma \in \Sigma$, $\sigma = (\vec{e}, \vec{m})$, $\vec{e} = \vec{m} = (1,0,0)$ as before: q^j_μ act on $H^{\otimes 5} \otimes H^{\otimes 5}$; if we let q_j, p_j denote Schroedinger's q, p acting on the j-th place in $H^{\otimes 5}$, we have

$$q^j_\mu = \begin{pmatrix} q_j \otimes I \\ p_j \otimes I \\ I \otimes q_j \\ I \otimes p_j \end{pmatrix},$$

and

$$A = \det \begin{pmatrix} I & q_1 \otimes I & p_1 \otimes I & I \otimes q_1 & I \otimes p_1 \\ I & q_2 \otimes I & p_2 \otimes I & I \otimes q_2 & I \otimes p_2 \\ - & - & - & - & - \\ I \cdot & q_5 \otimes I & p_5 \otimes I & I \otimes q_5 & I \otimes p_5 \end{pmatrix}$$

$$= \frac{1}{4} \sum_1^5 \varepsilon_{ijklm} M_{jk} \otimes M_{lm},$$

with $M_{jk} = q_j p_k - q_k p_j$ gen. of rotations in (j,k) plane in $SO(5)$,

$$[M_{jk}, M_{lm}] = i\big(\delta_{jl} M_{km} - \delta_{jm} M_{kl} + \\ + \delta_{km} M_{jl} - \delta_{kl} M_{jm}\big);$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \to \sqrt{5} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$A = \sqrt{5} \det \begin{pmatrix} 0 & q_1' \otimes I & p_1' \otimes I & I \otimes q_1' & I \times p_1' \\ 0 & q_2' \otimes I & p_2' \otimes I & I \otimes q_2' & I \otimes p_2' \\ 0 & \cdots & & \cdots & \\ 0 & \cdots & \cdots & \cdots & \\ I & q_5' \otimes I & p_5' \otimes I & I \otimes q_5' & I \otimes p_5' \end{pmatrix}$$

$$= \sqrt{5} \det \begin{pmatrix} q_1' \otimes I & p_1' \otimes I & I \otimes q_1' & I \otimes p_1' \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ q_4' \otimes I & p_4' \otimes I & I \otimes q_4' & I \otimes p_4' \end{pmatrix}$$

$= \sqrt{5} \cdot$ det (minors in the first 2 ~~rows~~ columns)

$\cdot$ det (compl. minor) $=$

UNCHANGED IF WE ROTATE TO ANOTHER ORTHONORMAL BASIS $\xi_1, \dots \xi_5$ IN $\mathbb{R}^5$: choosing

$$\xi_5 = \frac{1}{\sqrt{5}} (1,1,1,1,1)$$

we get $\longrightarrow$ 30'

$$A = \frac{1}{4} \sqrt{5} \sum_{1}^{4} \varepsilon_{ijkl} M'_{ij} \otimes M'_{kl} .$$

Now with

$$\vec{B} \equiv (M'_{23}, M'_{31}, M'_{12}),$$

$$\vec{D} \equiv (M'_{14}, M'_{24}, M'_{34}),$$

we have that

$$\vec{L}^{(\pm)} \equiv \frac{1}{2} (\vec{B} \pm \vec{D})$$

are mutually commuting generators of $SU(2)$ and

$$A = \sqrt{5} \left(\vec{L}^{(+)} \otimes \vec{L}^{(+)} + \vec{L}^{(-)} \otimes \vec{L}^{(-)} \right)$$

Now

$$\vec{J} \otimes \vec{J} = \frac{1}{2} \left\{ \left(\vec{J} \otimes I + I \otimes \vec{J} \right)^2 - \vec{J}^2 \otimes I - I \otimes \vec{J}^2 \right\}$$

by Clebsch-Gordan has eigenvalues

$$\frac{1}{2} \left(s(s+1) - u(u+1) - v(v+1) \right),$$

$$u, v \in \frac{1}{2} \mathbb{N}_0 = \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \right\}$$

$$s = u+v,\ u+v-1,\ \dots\ |u-v|;$$

keeping track of the fact that eigenvalues $s^+(s^+ +1)$ of $\vec{L}^{(+)2}$ and $s^-(s^- +1)$ of $\vec{L}^{(-)2}$ arising from reps of $SO(4)$ must be with s^+, s^- simultaneously integers or half integers, we see that

$$|dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4| \geq |A - 2q| \geq \sqrt{5} - 2.$$

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