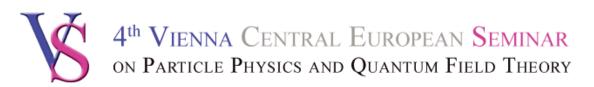
Faculty of Physics University of Vienna



November 30 - December 02, 2007

"COMMUTATIVE AND NONCOMMUTATIVE QUANTUM FIELDS"

Sergio Doplicher (Rome):

Quantum Spacetime and Noncommutative Geometry

Abstract:

After a survey on the status of Quantum Field Theory on Quantum Spacetime, recent results will be presented on the area and space volume operators, and especially on the Poincare' invariant spacetime volume operator, and on their spectral properties.

The BASIC MODEL OF QUANTUM SPACETIME IS THE SIMPLEST FULL-POINCARÉ (DVARIANT MODEL IMPLEMENTIC THE

SPACETIME UNCERTAINTY RELATIONS M Q.M. + (ce.) G.R.

DFR 1384, 1595

[9, 9,] = i Can

fulfilling the QUANT CONDITIONS:

=) THE DI'S should fat least obey:

 $\Delta q_{o} \sum_{j=1}^{s} \Delta q_{j} \gtrsim 1,$

$$\sum_{1 \le j < k \le 3} \Delta q_j \cdot \Delta q_k \gtrsim 1$$

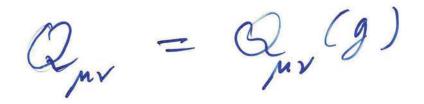
$$\sum_{1 \le j < k \le 3} \Delta q_j \cdot \Delta q_k \approx 1.6 \times 10^{-33} \text{ cm}$$

$$\left(1 = A_p^2 - A_p^2 \approx 1.6 \times 10^{-33} \text{ cm}\right)$$

UNCERTAINTY REL COMMUTATION RELIGIONS ?

$$[q_m, q_r] = i Q_{\mu r}$$

MORE CAREFUL ARGUMENT INDICASES



(WHERE THE PRECISE DEPENDENCE IS SOLL UNKNOWN), SO THROW

 $[q_m q_r] = i Q_{\mu r}(g)$

 $R_{mr} - \frac{1}{2} g_{mr} R = f_{mr} (2)$

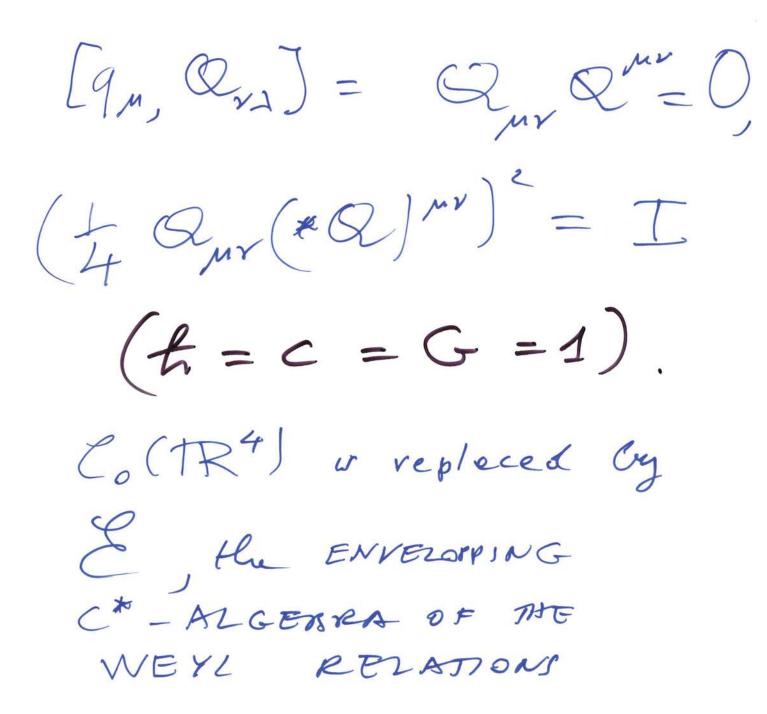
 $F_g(2/) = 0$

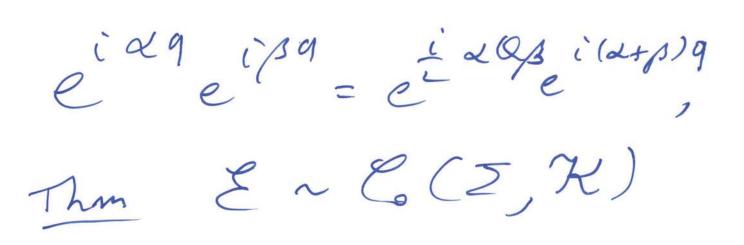
ME COUPLED EQ. OF MOTION: GEOMETRY ~ DYNAMICS, MGEBRA ~ DYNAMICS. RELATED TD:

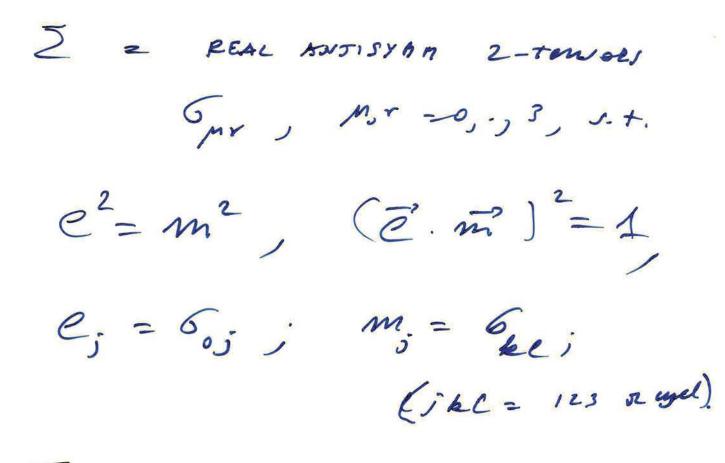
· COSMOLOGICAL CONSTANT = 0;

· ~ EQUILIBRIUM OF CMB WITHOUT INFLATION

S. D. ASXIV 2001, 2006.

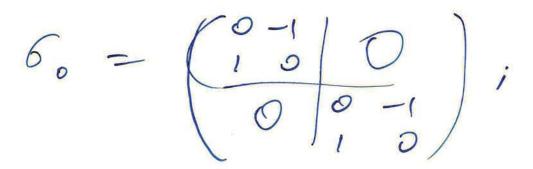






$\Sigma = \Sigma_{+} \cup \overline{\Sigma}_{-} =$

FULL LORENTZ ORBIT OF

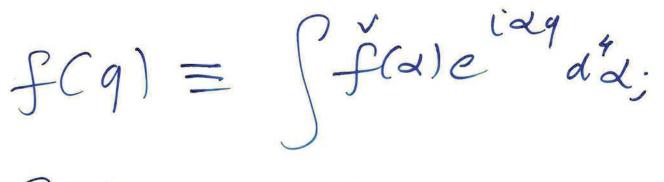


Corresponding REGULAR Mep: $q = \begin{pmatrix} Q \otimes I \\ P \otimes I \\ I \otimes Q \end{pmatrix}; Q, P = \\ I \otimes Q \end{pmatrix}; I - um.$ $I \otimes P \int CHROEDINGER$ O(S) IN THIS REP. :

2 9 = = 2(HOI + IOH) $H \equiv \frac{1}{2} \left(p^2 + Q^2 \right),$ $6(H) = \frac{1}{2} + N_{o;} \Rightarrow$ $\overline{Z} q_{\mu}^{2} \ge 2$ M = 2WHICH HOLDS IN ANY REP .: "=" PRECISELY IN REPS CHARLED BY 51, THE ISASE $26EZ, 6=(\bar{e},\bar{m}), \int_{0} 0 \neq \Sigma,$ $e^{2}=m^{2}=1$ = FULL ROTATION ORBIT OF 6. Z~TI_~ TS2 × 2=13.

INDER EVENTS : E & E & E & -- OzE Z = CENTRE M(E)~ ~ CB(Z); i.e. q;= I@...&q^@_...@_t, j-m. place so that $[q_{j}^{M}, q_{k}^{*}] = i \partial_{jk} Q$ INDER OF J. HEN Z (95-9xm > 4 M=O

Colenlus:



 $\int d^3q = Tr$

S d'g = lim Soly front. 90=t Squeixed d'a -> Seix.t g(20, 3) ky QFT :

 $\phi(q) = \int \tilde{\phi}(k) e^{ikq} d^{4}q$

FREE FIELDS - NON LOCAL BUT ROLMANE CON - LOCAL ALGEORAY, -

INTERACTION ?

MUST DEPINE INT. DESITY HCq), then $H_{I}(t) = \int d^{3}q H_{I}(q);$ do=t GELL-MADON LOW AND XJZ: S = TexpisH_(t) oft. DYSON EXPANSION: DFR 194-95: \$ interection $H_T(q) = g: \phi cq)^m$: STILL RE QUIRES ST. LORENTE INV. INSTE ANSAR $H_{I}(t) = \int dS \int d^{3}y H_{I}(q)$ $= \sum_{a} \int g_{a} = t$

MILD UV REG. (DFR) BUT SUPPICIENT FOR \$3(D.BAHNS). 2003

BUJ: : \$ (x): =

 $on \quad @ST: \quad \Sigma \quad (q_5 - q_k)^2 \ge 4!$

ALTERNATIVE :

 $H_{I}(q) = g E^{(m)} \phi(q_{1}) \dots \phi(q_{n})^{!}$ = g : $\phi^{2}_{cq} \phi(q_{n}) \dots \phi(q_{n})^{!}$ E^{m} = QUANTUM DIAGONAL MAP; $\mathcal{E}\otimes_{2} \cdot \otimes_{2} \mathcal{E} \longrightarrow \mathcal{E}$ ley: n times first in Iq; -> q @ I @ _ @ I; (m+1) fold @) them $q_j - q_n \longrightarrow I_Z \otimes (q_j - q_n)$ id & n @ . D M Evolute

where

 $M: Sf(Q, a) e^{idq} d^{n}a \longrightarrow$ \rightarrow $\int f(Q, \alpha) e^{-\frac{1}{4}|\chi|^2} d^{\frac{1}{4}} d^{\frac{1}{$ UNIVERSAL MAR S. T., IN REP. GNS OF W Z 9/m = 2 (= MIN.) w = Sdmon IPF for some prof. measure on I1. THUS THE Q. DIAG. MAP BRINGS 9; don to 9x AS MUCH AS KNOWED BY STUR. THE THEORY WITH $H_{I}(q) = q: p^{m}(q): Q$ UV - FINITE. IS WITH ADIADATIC DANDING

THAT IS, THE PERT EXP. OF < I, Sg(H) HI(H) at I)/ -00 VALUUR -> VACUUM UV-FIMTE. IS JERM BY TERM (D. B. ATTWS K. FRED PANDAR 6. MIDEISPLLI, S.D. 2903) HABBY ?NOT YET : BADLY BREAKS LORENTZ INV. g -> 1 15 A BROBLEM - AD. LIMIT (BVJ: KOSSOW). OTHER ABORDACHES: - YANG FELDMAN (BDFP, 2002) - QUASIFLAARE WICK PRODUCTS CBJF8,2005 + IN PREP.) ALL MEET (SOONER OR LATER! PROBLEMY

WITH LORGATZ INARISACE.

IS QST TO BE ABANDONED? NOV
- as LESS PROBLEMS THAN OTHER ATT.
- IMPOSED BY FIRST PRINCIPLES
- LEADS TO SUKPRISINGLY MCC BASIC GEOMETRY:
IN PARTICURE THE LORENTZ-IMARIANT 4 - STACETIME VOLUME OPERATOR
HAS A PURE DISCRETE SPECTRUM
WITH A GAP OF ORDER 1 (i.e. 24) FROM O. (S.D.K.FREDENHTSEEN in pres)

 $\frac{3}{2} |dq^{n}|^{2} \ge 4$; m=0 $Z \left[dq_{5} \wedge dq_{k} \right]^{2} \ge 1,$ \tilde{Z} $|Ag_{o}Ag_{j}|^{2} \geq 1;$ • $6\left(\left| dq^{1} \wedge dq^{2} \wedge dq^{3}\right|\right) = (0, +\infty)$ dyndyndyndy is A NORMAL OPERATOR WITH PURE ROINT SPECTRUM; 6 (dgn... ndg) = ±2+Zab+i(Za+Zb) $= \pm 2 + Z v_5 +$ 50. Khat + 2 (Z V5-2V5 + Z V5+2V5); $|dq_{1}...dq| \ge \sqrt{5} - 2.$ SPECTRUM WIJERE 2

2. NC DIFF CALCULUS

(completes with the min product ("-norm) each summand is a (C*-) olgebre mits own right. It is also an Ol - bimosule

A · (e, ⊗ · · ⊗ e_m) = ee, ⊗ · · ⊗ e_m
 (e, ⊗ · · ⊗ e_m) · b = a, ⊗ · · ⊗ e_m b,
 and w is Λ(O2). [MOTIVATION:
 We equip Λ(O2) with the
 O - bimedule tensor product:

 $(a_1 \otimes \cdots \otimes a_m)(b_1 \otimes \cdots \otimes b_m) \equiv$ $\Lambda_n(\Omega) \times \Lambda_m(\Omega) \rightarrow \Lambda_{m+m-1}(\Omega).$ Consider dA = Ixu-aQI, $a \in \Omega$ as element of $\Lambda(O2)$. D_{4} $\Omega(\Omega) \equiv d$ -stable suboly gen by Ω can be drown = n kee mk, where my is def on M, m > k, by: $m_{k}(e_{0}\otimes \ldots\otimes e_{n}) = e_{0}\otimes \ldots\otimes e_{k}\otimes \ldots\otimes e_{n}$ E Am. EN

Wont to define point $\Lambda(n) \times \Lambda(n) \longrightarrow \infty$ which give icalo product " on SLOD. "SHUFFLING PRODULT": < a, 0 - 00 an, b, 0 - 0 bm) $= \int_{mm}^{m} \prod_{i=1}^{m} a_i b_i$ Remark: let m=n=1, $\Lambda_{\lambda}(01) = 02, \langle a, b \rangle = ab.$ let m= n= 2, $da = I \otimes a - a \otimes I \in \Lambda_2(\alpha),$ $db = I \otimes b - b \otimes I \in \Lambda_2(\alpha),$ (lade) = ab .be - eb + eb [a,b]

More generally,

 $\langle a da_{1} \dots da_{m}, db_{n} \dots db_{m}, db_{n} \rangle =$ = $a \prod_{j=1}^{m} \Box a_{j}, b_{j}] b_{j}$

other expressions like

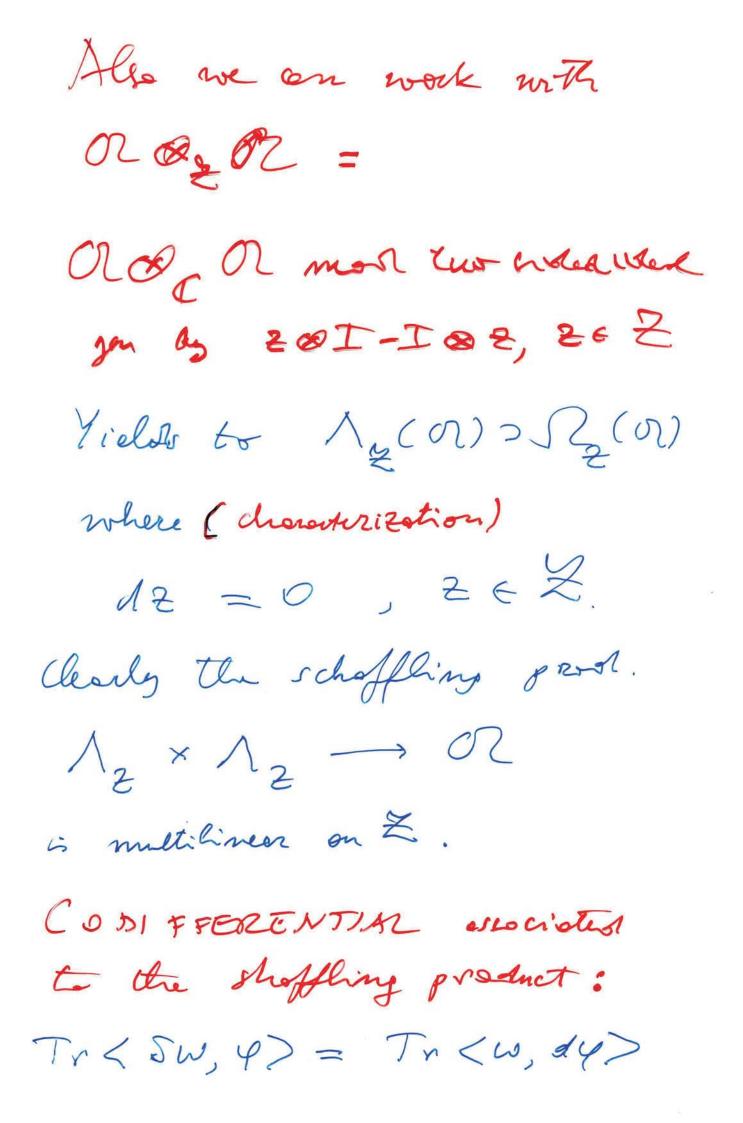
< dan den a, G. de, ... alm? we could computed by rejected use of

the Leibnitz rule

d(ab) = a db + (da) - b

Classially: combine pairing with INTEGRATION to get (...,) and the enounted TRANSPOSE S of A: S: nforms - n-1 forms; $dS+Sd = (d+SJ^2 = LAPLACIAN$ INTEGRATION -> TRACE on O

 $Tr: \Omega \rightarrow C$ Tr (ab) = Tr (be) (no pritivity reg.). Remork : Not necessary it is C-volued. The universal trace TR: M-> M ELIN SIAN EQ. B], a, b & OZ Z would dr. More concretely if Z = Centre O (Z. C. Centre of the MULTIPLIER ALGENRA OF A if A hed no unit) we can have 2-valued TRACE: $Tr: n \rightarrow 2$



PROP. The HOCHSCHILD BOUNDARY: S(a, @ ... @ em) = = Z(-1) ~ a, @ - @ a, R + , Ø - @ Q_m + (-1) and, Baz .. Ban is a costiflerential. * Good news end here. Bad news: the "quantum nove equation" $(dS+Fd) \varphi = 0$ does not seen of use for physics. BUT THERE IS A LOT OF GOOD USE FOR / (O) AND SCHAFFLING PRODUCT.

NOTE ME) has

two olg. structures:

. that of On- himos terms of . that of DE Zn ;

dq1... Noly has to be conjunted worth the first grod; it produces clements whose non spectra ... are arounded in the (C - completion) of the) second . Aprec: 5 = 1,2,3; $dq' \wedge dq' = (I aq' - q' aI)(I aq' - q' aI)$ $-(S \geq k) =$ $= I \otimes q' \otimes q' - I \otimes q' q' \otimes I - q' \otimes I \otimes q'$ + $y' \otimes g'' \otimes I - (j \xrightarrow{} k) =$ 5. Q. - i Q'K

here, as an spector in E 8282E, $\left| dq^{j} \wedge dq^{k} \right|^{2} = (s \cdot e \cdot)^{2} + (z \cdot x \cdot z)^{2}$ $\geq Q^{jk2}$

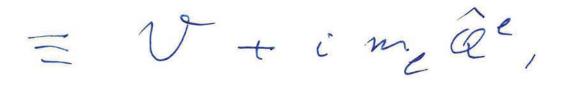
and

 $\left| dg^{j} dg^{k} \right|^{2} = \overline{m}^{2} \geq \overline{L};$ 5, k, e ujelie per \$ 1, 2, 3 finnlerly for the "timelike onee" $\sum |dg^{\circ} \wedge dg^{\circ}|^2 \neq \vec{e}^2 \geq T$ Space Volume : dgragnag = Eikk dgidgt dg =

= E (I@qz-q'OI)(I@qk-qtoI)(i@q'-q'oI)

= I terms with q's at 3 different tensor places (5.0.) +

+ I terms with 4/29e et one place and 9; et another (skew aborint by ontigemetasotion).





â	7	12	4 (-1) 9 5 2 (-1) 9 5	1
		2) = (/

with 9; = IO ... g @ ... I. sthe place

CLAIM: $O \in Sp | d\bar{q} \wedge d\bar{q} \wedge d\bar{q} | d\bar{q} |$. Suffice to work with intege $G \in \mathbb{Z}$. Choose $G = (\bar{e}, \bar{m}),$ with $\bar{m} = \bar{e} = (1, 0, 0);$

The onscioled regular imports on $H \otimes H$, H = L(R, ds) os $\begin{pmatrix} q \otimes I \\ p \otimes I \\ I \otimes q \end{pmatrix}$, $I \otimes p \end{pmatrix}$ $\begin{pmatrix} q^{\circ} \\ \vdots \\ q^{3} \end{pmatrix} =$

q = multipl by S, $p = -i\frac{d}{dc}$

We identify, by the obvious = (H @ H) @ 4 with H @ 4 @ H @ 4 Then, in the chosen irrep, $m_e \hat{Q}_e = m_1 \hat{Q}_1 = \frac{1}{2} Z(-1)^* p_j \otimes I_j$ $V = Z(-1)^{m+1} E_{jke} q_{1} - q_{4}$

-th place

 $= \begin{pmatrix} I & P_1 \otimes I & I \otimes q_1 & I \otimes P_1 \\ I & P_2 \otimes I & I \otimes q_2 & I \otimes P_2 \\ I & P_3 \otimes I & I \otimes q_3 & I \otimes P_3 \\ I & P_4 \otimes I & I \otimes q_4 & I \otimes P_4 \end{pmatrix}$

= Z E jken PR @ 9e Pm =

= $\frac{1}{4} \overline{2} \varepsilon^{ikem} (P_{e} - P_{ij}) \otimes M_{em}$

so that

 $d\overline{q} d\overline{q} d\overline{q} d\overline{q} = \frac{1}{2} Z C_{1}^{j+1} P_{j} \otimes I +$ + 1 ZEsken (Pt-P) & Men. Choose $x, y \in H^{604}, \|x\| = \|y\| = 1$, s.t. y E HODY = L2(TR4, d4s) dyrends upon I si only, and it has my in a sphere of reshing Ez around O; then || 1g ∧ 8g ∧ 8g × 8y || ≤ E.

Note: on any y with Amain of Men, Em=1,-,4, we have, it & 5 as before, 11 dg 1 dg 1 dg 2005 11 4 $\leq \mathcal{E}\left(1+\frac{1}{3}\sup||M_{en5}||\right).$

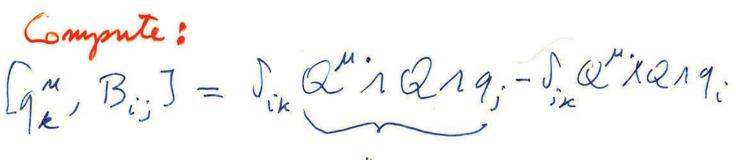
The Spacetime Volume Operate

dg Ady Ady Adg = = Emrap dg dg dg dg dg = = Enrip (I&gn-gnoI) ... (I&yr-groI) (products in Ag (E)!). = (no pair of g's nisome place) + S.Q. A''' (two peirs of g's nisome places) + S.Q. (two peirs of g's nisome places) + S.Q. --> Emrse I @ 9 9 0. I. @ 9 9 0 I = = 4 Emp I & [q", q"] & [& [q], y"] & [= $= -\frac{1}{4} Q \Lambda Q = -22, EZ,$ $\gamma = \pm I \quad on \quad \Sigma_{\pm}$.

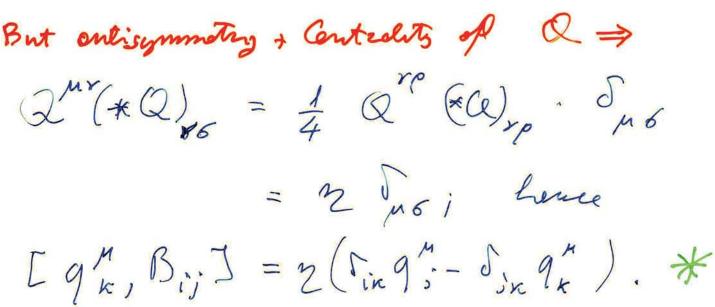
1gragragrag = A - 22 + its

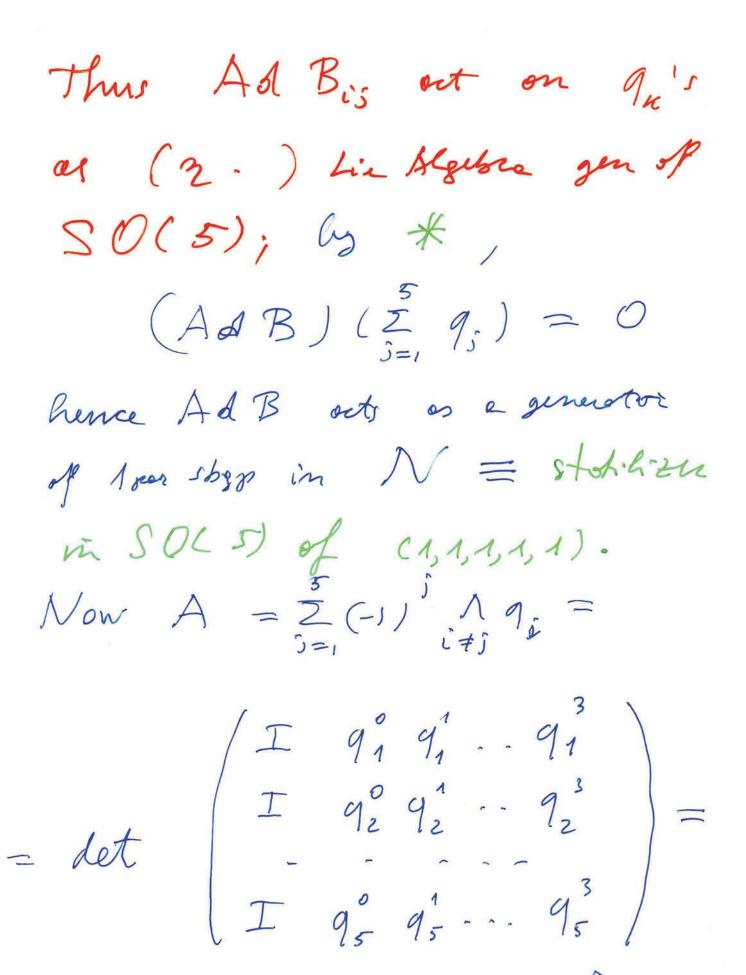
 $A = \sum_{j=1}^{5} (-1)^{j} \wedge q_{2} = \sum_{j=1}^{5} (-1)^{j} A_{j},$

 $B = \frac{1}{2} \sum_{i \neq j} (-1)^{i-j} Q_{\Lambda} q_{i} \Lambda q_{j} = \frac{1}{2} \overline{2} (-1)^{i-j} B_{j}.$



Erspo Qur Qdp 9; = $2(Q^{\mu r}(*Q)_{rs}) q_{s}^{6}$





= dot R. ())) and, if REN:

 $= det \begin{pmatrix} I & q'_{1}^{\prime 0} & q'_{1}^{\prime 1} & -- & q_{1}^{\prime 3} \\ I & q'_{2}^{\prime 0} & q'_{2}^{\prime 4} & -- & q'_{2}^{\prime 3} \\ -- & - & - & - \\ I & q'_{5}^{\prime 0} & q'_{5}^{\prime 2} & -- & q'_{5}^{\prime 3} \end{pmatrix}$ $q_{j}^{\prime m} = \mathcal{R}^{j \kappa} q_{\kappa}^{m},$ $\mathcal{R} \in \mathcal{N},$ ong generator and for 1- vor sage in N; of e $(A \circ D)(A) = O$

Home (A, B) = 0 and dgragnagnag= A-22+5B is NORMAL (2 is central!), and $|dy \wedge dy \wedge dy \wedge dy|^2 = (A - 22)^2 + TS^2$ $\geq (A - 22)^2$. Now as a field of operators on I, by LORENIZ imprime, dgragnagnagg à constant and of opporte signs on Z+ i I suffice to compute at GEZ, $6 = (\vec{e}, \vec{m}), \quad \vec{e} = \vec{m} = (1, 0, 0)$ or hefpre: q'' act on the & H & ; if we let q; p; danste Schroedinger's 9, p acting on the 3-th dace in HOS, we have *

 $q_{m}^{j} = \begin{pmatrix} q_{j} \otimes I \\ P_{j} \otimes I \\ I \otimes q_{j} \end{pmatrix}$

end

= 1 Z Eiskem M. OM with $M_{jk} = q_j P_k - q_k P_j$. gen. of rotations in (i,k) plane in SO(5) [M: Mem] = i (S. M - J. Mke + + Sem Mil - See Min);

 $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \end{pmatrix} \rightarrow \overline{V_5} \begin{pmatrix} P \\ 0 \\ 0 \\ 0 \\ 0 \\ \end{pmatrix} ,$

 $A = \sqrt{5} \text{ det} \begin{pmatrix} 0 & q'_{1} \otimes I & p'_{1} \otimes I & I \otimes q'_{1} & I \times p'_{1} \\ 0 & q'_{2} \otimes I & p'_{2} \otimes I & I \otimes q'_{2} & I \otimes p'_{2} \\ 0 & - & - & - \\ 0 & - & - & - \\ I & q'_{5} \otimes I & p'_{5} \otimes I & I \otimes q'_{5} & I \otimes p'_{5} \end{pmatrix}$ $= \cdot \sqrt{5} \det \begin{pmatrix} q_1' \otimes \overline{D} & p_1' \otimes \overline{D} & \overline{D} \otimes q_1' & \overline{D} \otimes p_1' \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & & \\ &$

det (minors nithe part 2 2005) = 15. - det Crongel-miner) -

UNCHANGES IF WE POTATE TO IN RS: choosing $S_{f} = \frac{1}{\sqrt{5}} (1, 1, 1, 1, 1, 1)$ ∋ 30' we get $\frac{1}{\sqrt{4}} = \frac{1}{4} \sqrt{5} \frac{2}{2} \mathcal{E}_{ijke} M'_{ij} \mathcal{M}_{ke}$ Now with $\vec{B} \equiv (M'_{23}, M'_{31}, M'_{12}),$ $\vec{D} \equiv (M'_{14}, M'_{24}, M'_{34}),$ we have that $\overline{L}^{(t)} \equiv \frac{4}{2} \left(\overrightarrow{B} \pm \overrightarrow{D} \right)$ ve mutually commuting generation of SU(2) and A = V5 (L'(+) @ L'(+) L'(-) @ L'(-))

Now

 $\vec{J} \otimes \vec{J} = \frac{1}{2} \left(\vec{J} \otimes I + I \otimes \vec{J} \right)^2$ $- \vec{J} \otimes I - I \otimes \vec{J}'$ by Clebsh - Groben has injervolnes $\frac{1}{2}(s(s+i) - n(n+i) - v(v+i)),$ u, v ∈ 2 A, = 292, 1, 2, -- 5 5 = u+v, u+v-1, -- [u-v]; heeping teck of the Jest that cipendues st(s+1) of ²(+)² our s⁻(s+1) of ²(-)² ²(-)² anny for eggs of SO(4)

must be not st, 5° sometonensly integers a holf integers, we see that

 $|dqndqndqndq| \geq |A-22| \geq V_5-2$.

REFERENCES

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