Metric interpretation of gauge fields in Noncommutative Geometry

P. Martinetti

Università di Roma, *La Sapienza*
Georg-August-Universität Göttingen

4th Central European Seminar on particle physics and quantum field theory,
Vienna, December 2007

Outline:

1. Distance in noncommutative geometry
   Connes formula

2. Gauge fields in NCG

3. Scalar fluctuation of the metric
   distance in the standard model

4. Gauge fluctuation of the metric
   holonomy obstruction
   explicit computation for a $U(2)$-bundle on S^1

5. On the line element in NCG
   Pythagore theorem ?

Conclusion

1. Distance in noncommutative geometry

A geometry "without points", but the notion of distance available via Connes formula. The metric information is encoded within the Dirac operator

$$ds = D^{-1}$$

1. Distance in noncommutative geometry

A geometry "without points", but the notion of distance available via Connes formula. The metric information is encoded within the Dirac operator

$$ds = D^{-1}$$

- Riemannian compact spin manifold M:

$$-i\gamma^\mu\partial_\mu \Longleftrightarrow \text{riemannian geodesic distance.}$$

1. Distance in noncommutative geometry

A geometry "without points", but the notion of distance available via Connes formula. The metric information is encoded within the Dirac operator

$$ds = D^{-1}$$

- Riemannian compact spin manifold M:

$$-i\gamma^\mu\partial_\mu \Longleftrightarrow \text{riemannian geodesic distance.}$$

- Fibre bundle P with connection:

$$-i\gamma^\mu(\partial_\mu + A_\mu) \Longleftrightarrow ?$$

1. Distance in noncommutative geometry

A geometry "without points", but the notion of distance available via Connes formula. The metric information is encoded within the Dirac operator

$$ds = D^{-1}$$

- Riemannian compact spin manifold M:

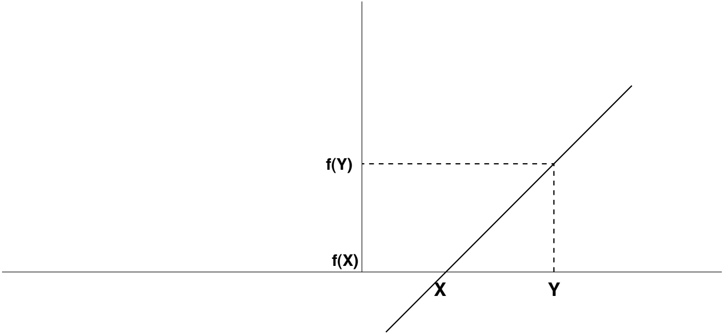
$$-i\gamma^\mu \partial_\mu \Longleftrightarrow \text{riemannian geodesic distance.}$$

- Fibre bundle P with connection:

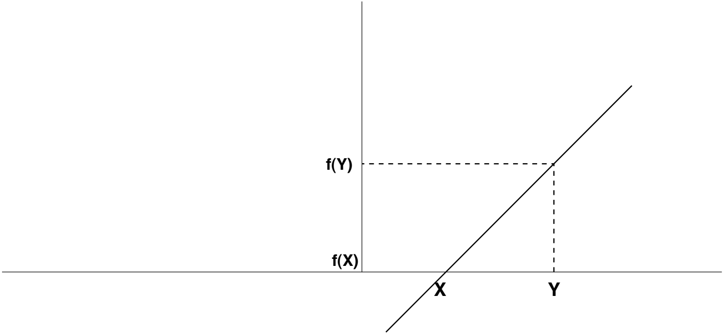
$$-i\gamma^\mu(\partial_\mu + A_\mu) \Longleftrightarrow ?$$

Noncommutative geometry provides a metric interpretation of the gauge fields, including the Higgs field *(Connes, Chamseddine, Lott)*.

Real line: $\sup_{f \in C^\infty(\mathbb{R})} \{|f(x) - f(y)| \,/\, \|f'\| \leq 1\} = |x - y|.$



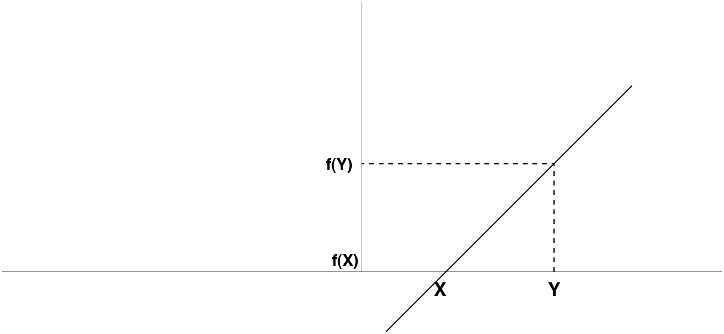
Real line: $\sup_{f \in C^\infty(\mathbb{R})} \{|f(x) - f(y)| \,/\, \|f'\| \leq 1\} = |x - y|.$



Riemannian spin manifold M:

$$\sup_{f \in C^\infty(M)} \{|f(x) - f(y)| \,/\, \left\| \overrightarrow{grad}\, f \right\| \leq 1\} = d_{geo}(x, y).$$

Real line: $\sup_{f \in C^\infty(\mathbb{R})} \{|f(x) - f(y)| \,/\, \|f'\| \leq 1\} = |x - y|.$



Riemannian spin manifold M:

$$\sup_{f \in C^\infty(M)} \{|f(x) - f(y)| \,/\, \left\| \overrightarrow{grad}\, f \right\| \leq 1\} = d_{geo}(x, y).$$

- The upper bound is attained because there exists f with $\left\| \overrightarrow{grad}\, f \right\| = 1$ everywhere on the geodesic (x, y), i.e $f(z) = d_{geo}(x, z)$.

Noncommutative space: by Gelfand duality,

$$\mathcal{P}(C^\infty(M)) \simeq M$$
$$\omega_x(f) = f(x)$$

with $\mathcal{P}(\mathcal{A})$ the pure states of $\mathcal{A}$ (normalized positive linear maps $C^\infty(M) \to \mathbb{C}$).

Noncommutative space: by Gelfand duality,

$$\mathcal{P}(C^\infty(M)) \simeq M$$
$$\omega_x(f) = f(x)$$

with $\mathcal{P}(\mathcal{A})$ the pure states of $\mathcal{A}$ (normalized positive linear maps $C^\infty(M) \to \mathbb{C}$).

$$\begin{aligned}
[\gamma^\mu \partial_\mu, f]\psi &= \gamma^\mu \partial_\mu f \psi - f \gamma^\mu \partial_\mu \psi = (\gamma^\mu \partial_\mu f)\psi \\
\|[\gamma^\mu \partial_\mu, f]\|^2 &= \|\gamma^\mu \partial_\mu f\|^2 = \|(\gamma^\mu \partial_\mu f)(\gamma^\nu \partial_\nu f)^*\| = \|g^{\mu\nu} \partial_\mu \partial_\nu f\| = \left\| \overrightarrow{grad}\, f \right\|
\end{aligned}$$

Noncommutative space: by Gelfand duality,

$$\mathcal{P}(C^{\infty}(M)) \simeq M$$
$$\omega_x(f) = f(x)$$

with $\mathcal{P}(\mathcal{A})$ the pure states of $\mathcal{A}$ (normalized positive linear maps $C^{\infty}(M) \to \mathbb{C}$).

$$\begin{aligned}
[\gamma^\mu \partial_\mu, f]\psi &= \gamma^\mu \partial_\mu f \psi - f \gamma^\mu \partial_\mu \psi = (\gamma^\mu \partial_\mu f)\psi \\
\left\|[\gamma^\mu \partial_\mu, f]\right\|^2 &= \left\|\gamma^\mu \partial_\mu f\right\|^2 = \|(\gamma^\mu \partial_\mu f)(\gamma^\nu \partial_\nu f)^*\| = \|g^{\mu\nu}\partial_\mu \partial_\nu f\| = \left\|\overrightarrow{grad}\, f\right\|
\end{aligned}$$

$$\boxed{d(\omega_x, \omega_y) \doteq \sup_{f \in C^{\infty}(M)} \left\{ |\omega_x(f) - \omega_y(f)| \,/\, \|[\gamma^\mu \partial_\mu, f]\| \leq 1 \right\}}$$

Noncommutative space: by Gelfand duality,

$$\mathcal{P}(C^\infty(M)) \simeq M$$
$$\omega_x(f) = f(x)$$

with $\mathcal{P}(\mathcal{A})$ the pure states of $\mathcal{A}$ (normalized positive linear maps $C^\infty(M) \to \mathbb{C}$).

$$\begin{aligned} [\gamma^\mu\partial_\mu, f]\psi &= \gamma^\mu\partial_\mu f\psi - f\gamma^\mu\partial_\mu\psi = (\gamma^\mu\partial_\mu f)\psi \\ \|[\gamma^\mu\partial_\mu, f]\|^2 &= \|\gamma^\mu\partial_\mu f\|^2 = \|(\gamma^\mu\partial_\mu f)(\gamma^\nu\partial_\nu f)^*\| = \|g^{\mu\nu}\partial_\mu\partial_\nu f\| = \left\|\overrightarrow{grad}\, f\right\| \end{aligned}$$

$$\boxed{d(\omega_x, \omega_y) \doteq \sup_{f\in C^\infty(M)} \{|\omega_x(f) - \omega_y(f)| \,/\, \|[\gamma^\mu\partial_\mu, f]\| \leq 1\}}$$

$\rightarrow$ Definition of the distance that still makes sense for noncommutative $\mathcal{A}$ as soon as there is a Dirac operator

$$\boxed{d(\omega_1, \omega_2) \doteq \sup_{a\in\mathcal{A}}\{|\omega_1(a) - \omega_2(a)| \,/\, \|[D, a]\| \leq 1\}}$$

Noncommutative space: by Gelfand duality,

$$\mathcal{P}(C^\infty(M)) \simeq M$$
$$\omega_x(f) = f(x)$$

with $\mathcal{P}(\mathcal{A})$ the pure states of $\mathcal{A}$ (normalized positive linear maps $C^\infty(M) \to \mathbb{C}$).

$$\begin{aligned}
[\gamma^\mu \partial_\mu, f]\psi &= \gamma^\mu \partial_\mu f \psi - f \gamma^\mu \partial_\mu \psi = (\gamma^\mu \partial_\mu f)\psi \\
\|[\gamma^\mu \partial_\mu, f]\|^2 &= \|\gamma^\mu \partial_\mu f\|^2 = \|(\gamma^\mu \partial_\mu f)(\gamma^\nu \partial_\nu f)^*\| = \|g^{\mu\nu} \partial_\mu \partial_\nu f\| = \left\| \overrightarrow{grad}\, f \right\|
\end{aligned}$$

$$\boxed{d(\omega_x, \omega_y) \doteq \sup_{f \in C^\infty(M)} \{|\omega_x(f) - \omega_y(f)| \,/\, \|[\gamma^\mu \partial_\mu, f]\| \leq 1\}}$$

→ Definition of the distance that still makes sense for noncommutative $\mathcal{A}$ as soon as there is a Dirac operator

$$\boxed{d(\omega_1, \omega_2) \doteq \sup_{a \in \mathcal{A}} \{|\omega_1(a) - \omega_2(a)| \,/\, \|[D, a]\| \leq 1\}}$$

→ coherent with the classical case when $\mathcal{A} = C^\infty(M)$: $d = d_{geo}$,

Noncommutative space: by Gelfand duality,

$$\mathcal{P}(C^\infty(M)) \simeq M$$
$$\omega_x(f) = f(x)$$

with $\mathcal{P}(\mathcal{A})$ the pure states of $\mathcal{A}$ (normalized positive linear maps $C^\infty(M) \to \mathbb{C}$).

$$\begin{aligned} [\gamma^\mu\partial_\mu, f]\psi &= \gamma^\mu\partial_\mu f\psi - f\gamma^\mu\partial_\mu\psi = (\gamma^\mu\partial_\mu f)\psi \\ \|[\gamma^\mu\partial_\mu, f]\|^2 &= \|\gamma^\mu\partial_\mu f\|^2 = \|(\gamma^\mu\partial_\mu f)(\gamma^\nu\partial_\nu f)^*\| = \|g^{\mu\nu}\partial_\mu\partial_\nu f\| = \left\|\overrightarrow{grad}\, f\right\| \end{aligned}$$

$$\boxed{d(\omega_x, \omega_y) \doteq \sup_{f \in C^\infty(M)} \{|\omega_x(f) - \omega_y(f)| \,/\, \|[\gamma^\mu\partial_\mu, f]\| \leq 1\}}$$

$\to$ Definition of the distance that still makes sense for noncommutative $\mathcal{A}$ as soon as there is a Dirac operator

$$\boxed{d(\omega_1, \omega_2) \doteq \sup_{a \in \mathcal{A}} \{|\omega_1(a) - \omega_2(a)| \,/\, \|[D, a]\| \leq 1\}}$$

$\to$ coherent with the classical case when $\mathcal{A} = C^\infty(M)$: $d = d_{geo}$,
$\to$ does not involve notions ill-defined in a quantum context (e.g. trajectories between points) but only spectral properties: *spectral distance*.

2. Gauge fields in NCG

2. Gauge fields in NCG

Connection: action of $In(A)$ on the geometry $(\mathcal{A} \xrightarrow{\pi} \mathcal{H}, D)$,

$$\pi \longrightarrow \pi \circ \alpha_u \Longleftrightarrow D \longrightarrow D_A \doteq D + A + JAJ^{-1}$$

$$A \doteq u[D, u^*].$$

2. Gauge fields in NCG

Connection: action of $In(A)$ on the geometry $(\mathcal{A} \xrightarrow{\pi} \mathcal{H}, D)$,

$$\pi \to \pi \circ \alpha_u \Longleftrightarrow D \to D_A \doteq D + A + JAJ^{-1}$$

$$A \doteq u[D, u^*].$$

Particular case of a *fluctuation of the metric* (export the metric to a Morita equivalent geometry; A characterizes the connection on the module):

$$A = \sum_i a_i[D, b_i].$$

2. Gauge fields in NCG

Connection: action of $In(A)$ on the geometry $(\mathcal{A} \xrightarrow{\pi} \mathcal{H}, D)$,

$$\pi \to \pi \circ \alpha_u \Longleftrightarrow D \to D_A \doteq D + A + JAJ^{-1}$$

$$A \doteq u[D, u^*].$$

Particular case of a *fluctuation of the metric* (export the metric to a Morita equivalent geometry; A characterizes the connection on the module):

$$A = \sum_i a_i[D, b_i].$$

Product of the continuum by the discrete:

2. Gauge fields in NCG

Connection: action of $In(A)$ on the geometry $(\mathcal{A} \xrightarrow{\pi} \mathcal{H}, D)$,

$$\pi \to \pi \circ \alpha_u \Longleftrightarrow D \to D_A \doteq D + A + JAJ^{-1}$$

$$A \doteq u[D, u^*].$$

Particular case of a *fluctuation of the metric* (export the metric to a Morita equivalent geometry; A characterizes the connection on the module):

$$A = \sum_i a_i [D, b_i].$$

Product of the continuum by the discrete:

$$\text{pure state:}(x, \omega_I) \Longleftarrow \quad \begin{array}{lcl} \mathcal{A} & = & C^\infty(M) \otimes \mathcal{A}_I \\ \mathcal{H} & = & L_2(M, S) \otimes \mathcal{H}_I \\ D & = & -i\gamma^\mu \partial_\mu \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \end{array}$$

2. Gauge fields in NCG

Connection: action of $In(A)$ on the geometry $(\mathcal{A} \xrightarrow{\pi} \mathcal{H}, D)$,

$$\pi \to \pi \circ \alpha_u \Longleftrightarrow D \to D_A \doteq D + A + JAJ^{-1}$$

$$A \doteq u[D, u^*].$$

Particular case of a *fluctuation of the metric* (export the metric to a Morita equivalent geometry; A characterizes the connection on the module):

$$A = \sum_i a_i[D, b_i].$$

Product of the continuum by the discrete:

$$\text{pure state:}(x, \omega_I) \Longleftarrow \begin{array}{lcl} \mathcal{A} & = & C^\infty(M) \otimes \mathcal{A}_I \\ \mathcal{H} & = & L_2(M, S) \otimes \mathcal{H}_I \\ D & = & -i\gamma^\mu \partial_\mu \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \end{array} \Longrightarrow A = H - i\gamma^\mu A_\mu$$

2. Gauge fields in NCG

Connection: action of $In(A)$ on the geometry $(\mathcal{A} \xrightarrow{\pi} \mathcal{H}, D)$,

$$\pi \to \pi \circ \alpha_u \Longleftrightarrow D \to D_A \doteq D + A + JAJ^{-1}$$

$$A \doteq u[D, u^*].$$

Particular case of a *fluctuation of the metric* (export the metric to a Morita equivalent geometry; A characterizes the connection on the module):

$$A = \sum_i a_i [D, b_i].$$

Product of the continuum by the discrete:

$$\text{pure state:}(x, \omega_I) \Longleftarrow \begin{array}{rcl} \mathcal{A} & = & C^\infty(M) \otimes \mathcal{A}_I \\ \mathcal{H} & = & L_2(M, S) \otimes \mathcal{H}_I \\ D & = & -i\gamma^\mu \partial_\mu \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \end{array} \Longrightarrow A = H - i\gamma^\mu A_\mu$$

- H: scalar field on M with value in $\mathcal{A}_I$ $\quad\to$ Higgs.

2. Gauge fields in NCG

Connection: action of $In(A)$ on the geometry $(\mathcal{A} \xrightarrow{\pi} \mathcal{H}, D)$,

$$\pi \to \pi \circ \alpha_u \Longleftrightarrow D \to D_A \doteq D + A + JAJ^{-1}$$

$$A \doteq u[D, u^*].$$

Particular case of a *fluctuation of the metric* (export the metric to a Morita equivalent geometry; A characterizes the connection on the module):

$$A = \sum_i a_i [D, b_i].$$

Product of the continuum by the discrete:

$$\text{pure state:}(x, \omega_I) \Longleftarrow \begin{array}{rcl} \mathcal{A} & = & C^\infty(M) \otimes \mathcal{A}_I \\ \mathcal{H} & = & L_2(M, S) \otimes \mathcal{H}_I \\ D & = & -i\gamma^\mu \partial_\mu \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \end{array} \Longrightarrow A = H - i\gamma^\mu A_\mu$$

- H: scalar field on M with value in $\mathcal{A}_I$ $\to$ Higgs.
- A_μ: 1-form field with value in $Lie(U(\mathcal{A}_I))$ $\to$ gauge field.

3. Scalar fuctuation $A_\mu = 0, H \neq 0$ *(Wulkenhaar, P.M. 2001)*

3. Scalar fuctuation $A_\mu = 0, H \neq 0$ *(Wulkenhaar, P.M. 2001)*

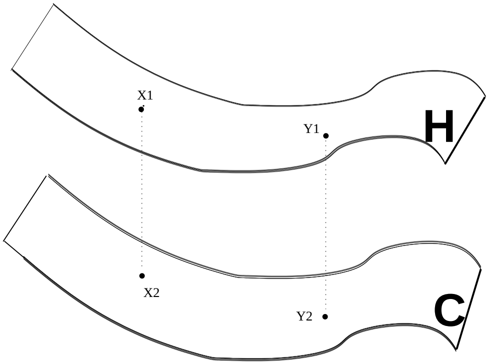
$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I$ with $\mathcal{A}_I = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

3. Scalar fuctuation $A_\mu = 0, H \neq 0$ *(Wulkenhaar, P.M. 2001)*

$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I$ with $\mathcal{A}_I = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \Longrightarrow \mathcal{P}(\mathcal{A})$ is a two-sheet model

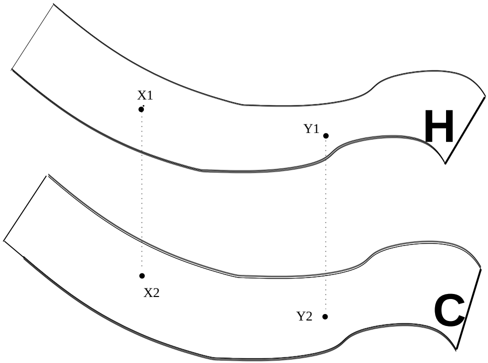
3. Scalar fuctuation $A_\mu = 0, H \neq 0$ *(Wulkenhaar, P.M. 2001)*

$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I$ with $\mathcal{A}_I = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \Longrightarrow \mathcal{P}(\mathcal{A})$ is a two-sheet model



3. Scalar fuctuation $A_\mu = 0, H \neq 0$ *(Wulkenhaar, P.M. 2001)*

$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I$ with $\mathcal{A}_I = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \Longrightarrow \mathcal{P}(\mathcal{A})$ is a two-sheet model



The spectral distance d coincides with the geodesic distance in $M \times [0,1]$ given by

$$\begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \left(|1+h_1|^2 + |h_2|^2\right) m_{\text{top}}^2 \end{pmatrix} \text{ where } \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \text{ is the Higgs doublet.}$$

4. Gauge fluctuation $A_\mu \neq 0, H = 0$

4. Gauge fluctuation $A_\mu \neq 0, H = 0$

Example suggested by Connes (96)

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I \text{ with } \mathcal{A}_I = M_n(\mathbb{C}),\ D_I = 0.$$

4. Gauge fluctuation $A_\mu \neq 0, H = 0$

Example suggested by Connes (96)

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I \text{ with } \mathcal{A}_I = M_n(\mathbb{C}),\ D_I = 0.$$

$\mathcal{P}(\mathcal{A})$ is a trivial bundle $P \xrightarrow{\pi} M$ with fiber $\mathbb{C}P^{n-1}$,

$$P \ni p = (x, \xi) = \xi_x, \quad \xi_x(a) = \langle \xi, a(x)\xi \rangle = \mathrm{Tr}(s_\xi a(x)).$$

4. Gauge fluctuation $A_\mu \neq 0, H = 0$

Example suggested by Connes (96)

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_I \text{ with } \mathcal{A}_I = M_n(\mathbb{C}),\ D_I = 0.$$

$\mathcal{P}(\mathcal{A})$ is a trivial bundle $P \xrightarrow{\pi} M$ with fiber $\mathbb{C}P^{n-1}$,

$$P \ni p = (x, \xi) = \xi_x, \quad \xi_x(a) = \langle \xi, a(x)\xi \rangle = \mathrm{Tr}(s_\xi a(x)).$$

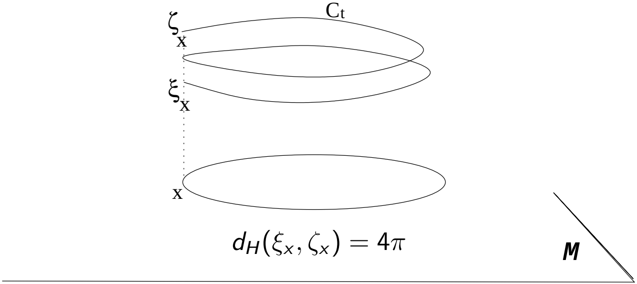
The part of D_A that does not commute with the representation is the covariant Dirac operator $-i\gamma^\mu(\partial_\mu + A_\mu)$ associated to to the connection.

The connection defines both a spectral distance d and an horizontal distance d_H:

$$T_pP = V_pP \oplus H_pP \Longrightarrow d_H(p,q) = \operatorname*{Inf}_{\dot{c}_t \in H_{c_t}P} \int_0^1 \|\dot{c}_t\| \, dt.$$

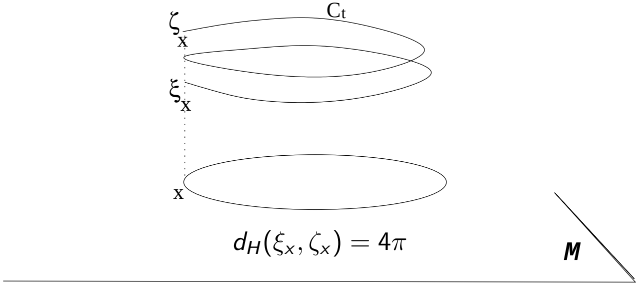
The connection defines both a spectral distance d and an horizontal distance d_H:

$$T_pP = V_pP \oplus H_pP \Longrightarrow d_H(p,q) = \operatorname*{Inf}_{\dot{c}_t \in H_{c_t}P} \int_0^1 \|\dot{c}_t\|\, dt.$$



The connection defines both a spectral distance d and an horizontal distance d_H:

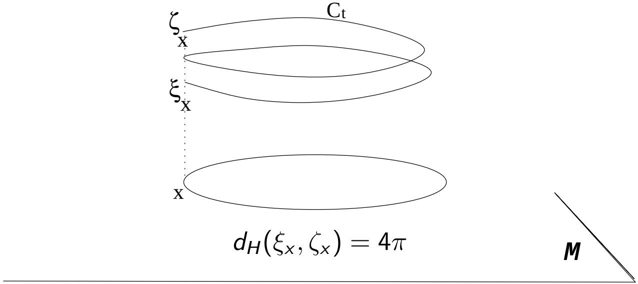
$$T_pP = V_pP \oplus H_pP \Longrightarrow d_H(p,q) = \operatorname*{Inf}_{\dot{c}_t \in H_{c_t}P} \int_0^1 \|\dot{c}_t\|\, dt.$$



$$d \leq d_H$$

The connection defines both a spectral distance d and an horizontal distance d_H:

$$T_pP = V_pP \oplus H_pP \Longrightarrow d_H(p,q) = \underset{\dot{c}_t \in H_{c_t}P}{\text{Inf}} \int_0^1 \|\dot{c}_t\| \, dt.$$



$$d_H(\xi_x, \zeta_x) = 4\pi$$

$$d \leq d_H$$

points at finite horizontal distance $\swarrow$ $\text{Acc}(\xi_x)$

points at finite spectral distance $\searrow$ $\text{Con}(\xi_x)$

$$\boxed{\text{Acc}(\xi_x) \subset \text{Con}(\xi_x)}$$

Holonomy obstruction

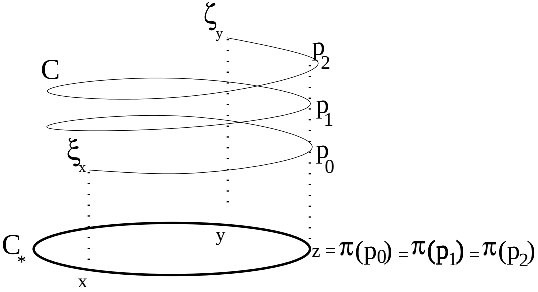
$$f(z) = \omega_z(f) = d_{\mathrm{geo}}(x, z) \Longrightarrow C_t(a) = d_H(\xi_x, C_t)$$

for any C_t in the minimal horizontal curve C between $\xi_x = C_0$, $\zeta_y = C_1$.

Holonomy obstruction

$$f(z) = \omega_z(f) = d_{\text{geo}}(x, z) \Longrightarrow C_t(a) = d_H(\xi_x, C_t)$$

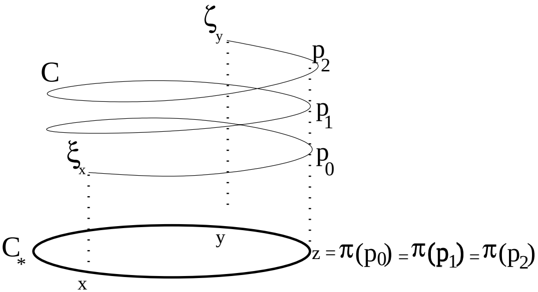
for any C_t in the minimal horizontal curve C between $\xi_x = C_0$, $\zeta_y = C_1$.



Holonomy obstruction

$$f(z) = \omega_z(f) = d_{\text{geo}}(x, z) \Longrightarrow C_t(a) = d_H(\xi_x, C_t)$$

for any C_t in the minimal horizontal curve C between $\xi_x = C_0$, $\zeta_y = C_1$.



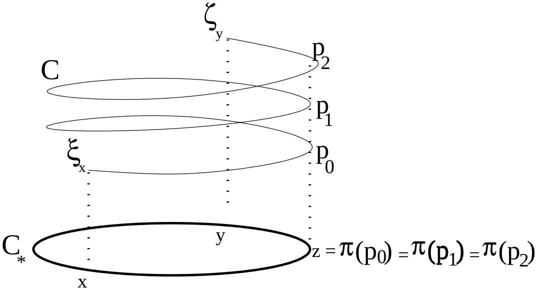
For $C_t = p_i, i = 0, 1, 2, \dots k$,

$$\text{Tr}(s_{p_i} a(z)) = d_H(\xi_x, p_i).$$

Holonomy obstruction

$$f(z) = \omega_z(f) = d_{\text{geo}}(x,z) \Longrightarrow C_t(a) = d_H(\xi_x, C_t)$$

for any C_t in the minimal horizontal curve C between $\xi_x = C_0$, $\zeta_y = C_1$.



For $C_t = p_i, i = 0, 1, 2, \ldots k$,

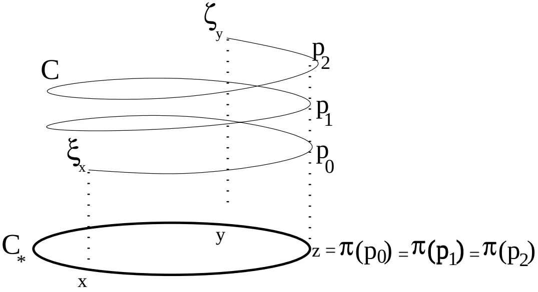
$$\text{Tr}(s_{p_i} a(z)) = d_H(\xi_x, p_i).$$

- If $k > n^2$, too many conditions on $a(z)$! The spectral and horizontal distances cannot be equal.

Holonomy obstruction

$$f(z) = \omega_z(f) = d_{\text{geo}}(x, z) \Longrightarrow C_t(a) = d_H(\xi_x, C_t)$$

for any C_t in the minimal horizontal curve C between $\xi_x = C_0$, $\zeta_y = C_1$.



For $C_t = p_i, i = 0, 1, 2, \dots k$,

$$\text{Tr}(s_{p_i} a(z)) = d_H(\xi_x, p_i).$$

- If $k > n^2$, too many conditions on $a(z)$! The spectral and horizontal distances cannot be equal.
- Can one find a minimal horizontal curve with less than n^2 intersecting points ?

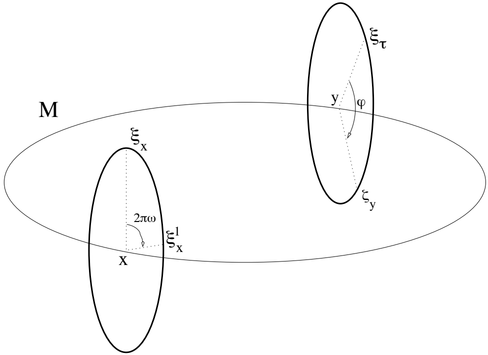
Spectral distance on the circle

Spectral distance on the circle

$\mathcal{A} = C^\infty(S^1, M_n(\mathbb{C})) \Longrightarrow$ pure states form a $\mathbb{C}P^{n-1}$ trivial bundle on S^1,

Spectral distance on the circle

$\mathcal{A} = C^\infty(S^1, M_n(\mathbb{C})) \Longrightarrow$ pure states form a $\mathbb{C}P^{n-1}$ trivial bundle on S^1,



$$\omega_j \doteq \int_0^{2\pi} \frac{\theta_1(t) - \theta_j(t)\, dt}{2},\ A = i \begin{pmatrix} \theta_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \theta_n \end{pmatrix},\ \xi_x = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} \in \mathbb{C}P^{n-1}.$$

Fiberwise,

$$\mathrm{Acc}(\xi_x) \cap \pi^{-1}(x) = \begin{pmatrix} V_1 \\ e^{2i\pi k\omega_j} V_j \end{pmatrix},\ k \in \mathbb{Z},\ j = 2, \ldots, n.$$

Fiberwise,

$$\mathrm{Acc}(\xi_x) \cap \pi^{-1}(x) = \begin{pmatrix} V_1 \\ e^{2i\pi k\omega_j} V_j \end{pmatrix},\ k \in \mathbb{Z},\ j = 2, ..., n.$$

$$\mathrm{Con}(\xi_x) \cap \pi^{-1}(x) = \begin{pmatrix} V_i & \forall i \in \mathrm{Far}_1 \\ e^{i\varphi_2} V_i & \forall i \in \mathrm{Far}_2 \\ \dots & \\ e^{i\varphi_{n_c}} V_i & \forall i \in \mathrm{Far}_{n_c} \end{pmatrix}, \varphi_j \in \mathbb{R},\ j \in [2, n_c]$$

where Far_j are the classes of equivalence of $i \sim j$ iff $\omega_j = \omega_i \bmod [2\pi]$.

Fiberwise,

$$\mathrm{Acc}(\xi_x) \cap \pi^{-1}(x) = \begin{pmatrix} V_1 \\ e^{2i\pi k\omega_j} V_j \end{pmatrix},\ k \in \mathbb{Z},\ j = 2, \ldots, n.$$

$$\mathrm{Con}(\xi_x) \cap \pi^{-1}(x) = \begin{pmatrix} V_i & \forall i \in \mathrm{Far}_1 \\ e^{i\varphi_2} V_i & \forall i \in \mathrm{Far}_2 \\ \ldots & \\ e^{i\varphi_{n_c}} V_i & \forall i \in \mathrm{Far}_{n_c} \end{pmatrix}, \varphi_j \in \mathbb{R},\ j \in [2, n_c]$$

where Far_j are the classes of equivalence of $i \sim j$ iff $\omega_j = \omega_i$ mod$[2\pi]$.

- $\mathrm{Con}(\xi_x)$ is a torus inside $\mathbb{C}P^{n-1}$ with dimension n_c given by the holonomy.

Fiberwise,

$$\mathrm{Acc}(\xi_x) \cap \pi^{-1}(x) = \begin{pmatrix} V_1 \\ e^{2i\pi k\omega_j} V_j \end{pmatrix}, \; k \in \mathbb{Z}, \; j = 2, \ldots, n.$$

$$\mathrm{Con}(\xi_x) \cap \pi^{-1}(x) = \begin{pmatrix} V_i & \forall i \in \mathrm{Far}_1 \\ e^{i\varphi_2} V_i & \forall i \in \mathrm{Far}_2 \\ \cdots & \\ e^{i\varphi_{n_c}} V_i & \forall i \in \mathrm{Far}_{n_c} \end{pmatrix}, \varphi_j \in \mathbb{R}, \; j \in [2, n_c]$$

where Far_j are the classes of equivalence of $i \sim j$ iff $\omega_j = \omega_i \bmod[2\pi]$.

- $\mathrm{Con}(\xi_x)$ is a torus inside $\mathbb{C}P^{n-1}$ with dimension n_c given by the holonomy.
- $\mathrm{Acc}(\xi_x)$ is the orbit of ξ_x under the action of the holonomy group.
  At best it is dense within $\mathrm{Con}(\xi_x)$.

Fiberwise,

$$\mathrm{Acc}(\xi_x) \cap \pi^{-1}(x) = \begin{pmatrix} V_1 \\ e^{2i\pi k\omega_j} V_j \end{pmatrix}, \; k \in \mathbb{Z}, \; j = 2, \ldots, n.$$

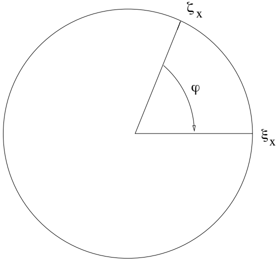
$$\mathrm{Con}(\xi_x) \cap \pi^{-1}(x) = \begin{pmatrix} V_i & \forall i \in \mathrm{Far}_1 \\ e^{i\varphi_2} V_i & \forall i \in \mathrm{Far}_2 \\ \cdots & \\ e^{i\varphi_{n_c}} V_i & \forall i \in \mathrm{Far}_{n_c} \end{pmatrix}, \varphi_j \in \mathbb{R}, \; j \in [2, n_c]$$

where Far_j are the classes of equivalence of $i \sim j$ iff $\omega_j = \omega_i \bmod[2\pi]$.

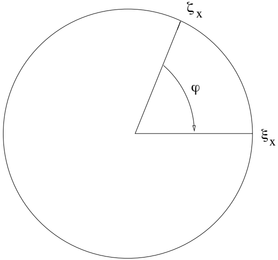
- $\mathrm{Con}(\xi_x)$ is a torus inside $\mathbb{C}P^{n-1}$ with dimension n_c given by the holonomy.
- $\mathrm{Acc}(\xi_x)$ is the orbit of ξ_x under the action of the holonomy group. At best it is dense within $\mathrm{Con}(\xi_x)$.

$$\boxed{\mathrm{Acc}(\xi_x) \subsetneq \mathrm{Con}(\xi_x)}$$

The shape of the fiber for $n = 2$

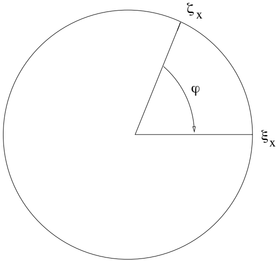


The shape of the fiber for $n = 2$



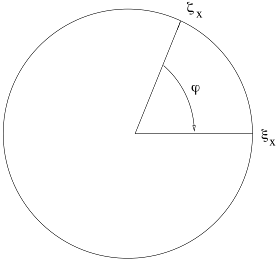
$\Bigg\{$

The shape of the fiber for $n = 2$



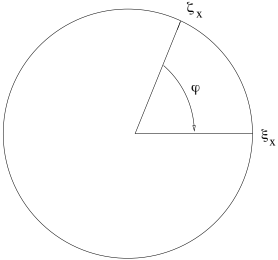
$$\begin{cases} d_H(0,\varphi) = 2k\pi \qquad \text{if} \qquad \varphi = 2k\pi\omega \mod [2\pi] \\ \end{cases}$$

The shape of the fiber for $n = 2$

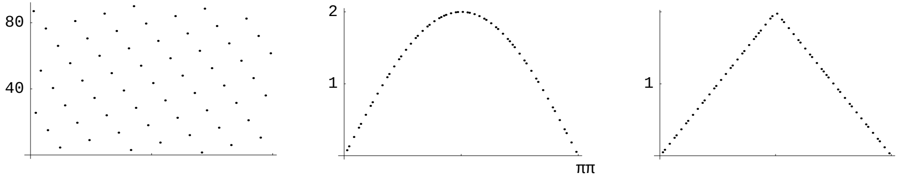


$$\begin{cases} d_H(0,\varphi) = 2k\pi & \text{if} \quad \varphi = 2k\pi\omega \mod [2\pi] \\ d(0,\varphi) = C \sin\frac{\varphi}{2} & \text{with} \quad C = \frac{4\pi|V_1||V_2|}{|\sin\omega\pi|} \end{cases}$$

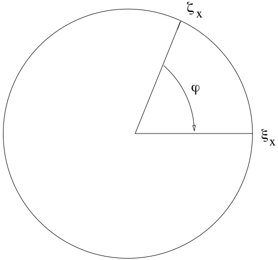
The shape of the fiber for $n = 2$



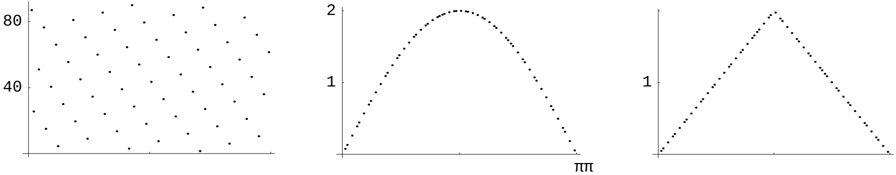
$$\begin{cases} d_H(0,\varphi) = 2k\pi & \text{if} \quad \varphi = 2k\pi\omega \mod [2\pi] \\ d(0,\varphi) = C\sin\frac{\varphi}{2} & \text{with} \quad C = \frac{4\pi|V_1||V_2|}{|\sin\omega\pi|} \end{cases}$$



The shape of the fiber for $n = 2$

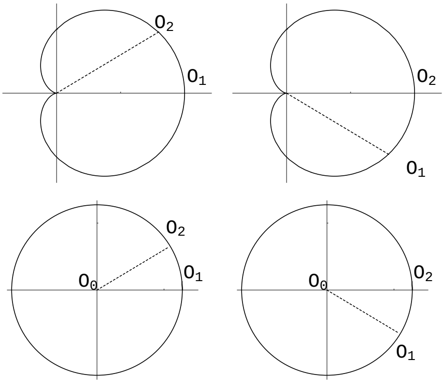


$$\begin{cases} d_H(0,\varphi) = 2k\pi & \text{if} \quad \varphi = 2k\pi\omega \mod [2\pi] \\ d(0,\varphi) = C\sin\frac{\varphi}{2} & \text{with} \quad C = \frac{4\pi|V_1||V_2|}{|\sin\omega\pi|} \end{cases}$$

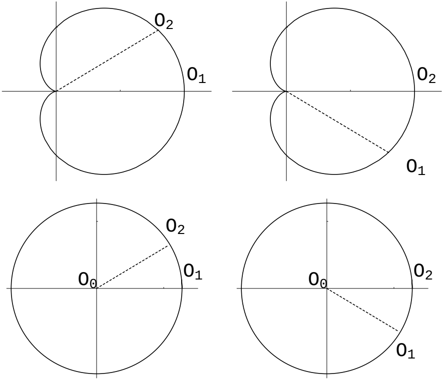


- No cutlocus for the distance function d: the fiber is smoother than a circle.

First interpretation: $d(0,\varphi)$ is the euclidean distance on the cardioid. But the latest is not invariant by rotation whereas d is.

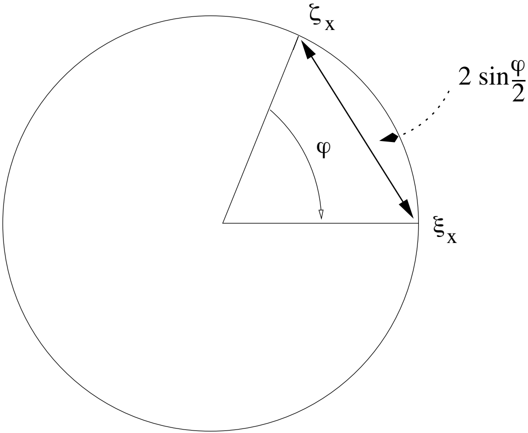


First interpretation: $d(0, \varphi)$ is the euclidean distance on the cardioid. But the latest is not invariant by rotation whereas d is.

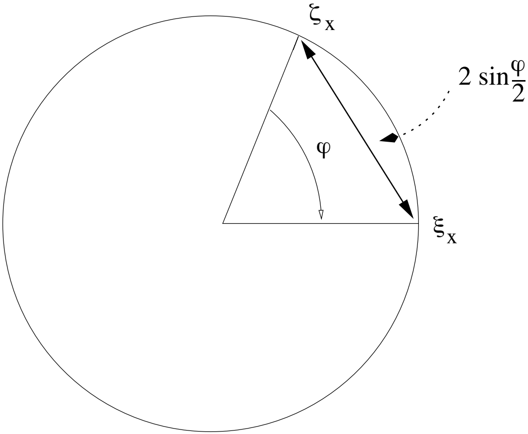


- With the spectral distance, everyone can equally pretend to be the center of the world.

Second interpretation: length of the segment in the disk

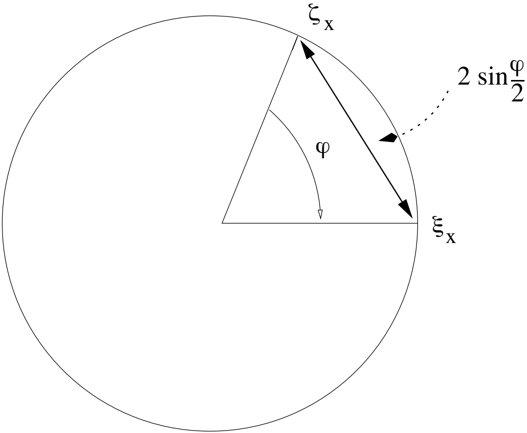


Second interpretation: length of the segment in the disk



The spectral distance “sees” the disk through the circle, in the same way as it sees between the sheets of the standard model.

Second interpretation: length of the segment in the disk



The spectral distance "sees" the disk through the circle, in the same way as it sees between the sheets of the standard model.

- The spectral distance "sees" the disk through the circle, in the same way as it sees between the sheets of the standard model.

5. On the line element in noncommutative geometry

$$ds = D^{-1}$$

5. On the line element in noncommutative geometry

$$ds = D^{-1}$$

For the product of geometry,

$$\begin{aligned} D = -i\gamma^\mu \partial_\mu \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \quad &\Longrightarrow \quad D^2 = (\gamma^\mu \partial_\mu)^2 \otimes \mathbb{I}_I + \mathbb{I}_E \otimes D_I^2 \\ &\Longrightarrow \quad ds^{-2} = ds_E^{-2} + ds_I^{-2}. \end{aligned}$$

5. On the line element in noncommutative geometry

$$ds = D^{-1}$$

For the product of geometry,

$$D = -i\gamma^\mu \partial_\mu \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \quad \Longrightarrow \quad D^2 = (\gamma^\mu \partial_\mu)^2 \otimes \mathbb{I}_I + \mathbb{I}_E \otimes D_I^2$$
$$\Longrightarrow \quad ds^{-2} = ds_E^{-2} + ds_I^{-2}.$$

- Pythagore^{-1} (cf Connes' post on NCG blog).

5. On the line element in noncommutative geometry

$$ds = D^{-1}$$

For the product of geometry,

$$\begin{aligned} D = -i\gamma^\mu\partial_\mu \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \quad &\Longrightarrow \quad D^2 = (\gamma^\mu\partial_\mu)^2 \otimes \mathbb{I}_I + \mathbb{I}_E \otimes D_I^2 \\ &\Longrightarrow \quad ds^{-2} = ds_E^{-2} + ds_I^{-2}. \end{aligned}$$

- Pythagore^{-1} (cf Connes' post on NCG blog).

Two-sheet model,

$$\begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \left(|1+h_1|^2 + |h_2|^2\right) m_{\text{top}}^2 \end{pmatrix} \Longrightarrow ds^2 = ds_E^2 + ds_I^2.$$

5. On the line element in noncommutative geometry

$$ds = D^{-1}$$

For the product of geometry,

$$D = -i\gamma^\mu \partial_\mu \otimes \mathbb{I}_I + \gamma^5 \otimes D_I \quad \Longrightarrow \quad D^2 = (\gamma^\mu \partial_\mu)^2 \otimes \mathbb{I}_I + \mathbb{I}_E \otimes D_I^2$$
$$\Longrightarrow \quad ds^{-2} = ds_E^{-2} + ds_I^{-2}.$$

- Pythagore^{-1} (cf Connes' post on NCG blog).

Two-sheet model,

$$\begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \left(|1+h_1|^2 + |h_2|^2\right) m_{\text{top}}^2 \end{pmatrix} \Longrightarrow ds^2 = ds_E^2 + ds_I^2.$$

- What is the equivalent for the disk ?

$$ds^2_{\text{disk}} = \text{function}(ds^2_{\text{circle}}, A) \ ?$$

Conclusion

Conclusion

- Distance on the fiber for $n \geq 2$, $d(\xi_x, \zeta_x) = \pi \mathrm{Tr}|S|$ where S is the matrix with components

$$S_{ij} \doteq 2|V_i||V_j| \, \frac{\sin\left(\frac{\varphi_j - \varphi_i}{2}\right)}{\sin \pi(\omega_j - \omega_i)}.$$

Conclusion

- Distance on the fiber for $n \geq 2$, $d(\xi_x, \zeta_x) = \pi \mathrm{Tr}|S|$ where S is the matrix with components

$$S_{ij} \doteq 2|V_i||V_j| \frac{\sin\left(\frac{\varphi_j - \varphi_i}{2}\right)}{\sin \pi(\omega_j - \omega_i)}.$$

- Formula for distance between the fibers.

Conclusion

- Distance on the fiber for $n \geq 2$, $d(\xi_x, \zeta_x) = \pi \text{Tr}|S|$ where S is the matrix with components

$$S_{ij} \doteq 2|V_i||V_j| \frac{\sin\left(\frac{\varphi_j - \varphi_i}{2}\right)}{\sin \pi(\omega_j - \omega_i)}.$$

- Formula for distance between the fibers.

- From a bundle point of view, the spectral distance seems more interesting than the horizontal one since it sees through the leaves of the foliation.

Conclusion

- Distance on the fiber for $n \geq 2$, $d(\xi_x, \zeta_x) = \pi \text{Tr}|S|$ where S is the matrix with components

$$S_{ij} \doteq 2|V_i||V_j| \, \frac{\sin\left(\frac{\varphi_j - \varphi_i}{2}\right)}{\sin \pi(\omega_j - \omega_i)}.$$

- Formula for distance between the fibers.

- From a bundle point of view, the spectral distance seems more interesting than the horizontal one since it sees through the leaves of the foliation.

- Topological effect due to $M = S^1$? Other basis requires to know the number of selfintersecting points of C_*.

Conclusion

- Distance on the fiber for $n \geq 2$, $d(\xi_x, \zeta_x) = \pi \mathrm{Tr}|S|$ where S is the matrix with components
$$S_{ij} \doteq 2|V_i||V_j| \, \frac{\sin\left(\frac{\varphi_j - \varphi_i}{2}\right)}{\sin \pi(\omega_j - \omega_i)}.$$

- Formula for distance between the fibers.

- From a bundle point of view, the spectral distance seems more interesting than the horizontal one since it sees through the leaves of the foliation.

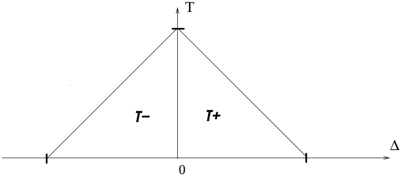
- Topological effect due to $M = S^1$? Other basis requires to know the number of selfintersecting points of C_*.

Spectral distance on S^1: math.OA/0703586, submitted to J. Func. Anal.
CC vs NC-distance: Com.Math.Phys. **265** (2006) 585-616,
or a non technical version, Cluj university press, *hep-th/0603051*.
Scalar fluctuation: with R. Wulkenhaar, J.Math.Phys. **43** (2002) 182-204.

Distance between the fibers:

Distance between the fibers:

$$d(\xi_x, \zeta_y) = \max_{\mathcal{T}_\pm} H_\xi(T, \Delta)$$



unit is $\min(\tau, 2\pi - \tau)$

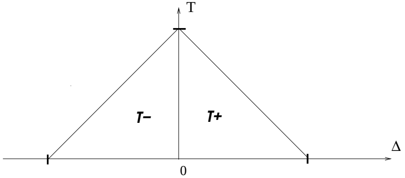
where the sign is the one of $z_\xi \doteq V_1^2 - V_2^2$,

$$H_\xi(T, \Delta) \doteq T + z_\xi \Delta + W_1 \sqrt{(\tau - T)^2 - \Delta^2} + W_0 \sqrt{(2\pi - \tau - T)^2 - \Delta^2}$$

$$W_0 \doteq R \frac{|\sin(\frac{\varphi}{2})|}{|\sin \omega\pi|}, \; W_1 \doteq R \frac{|\sin(\omega\pi + \frac{\varphi}{2})|}{|\sin \omega\pi|}, \; R \doteq \sqrt{1 - z_\xi^2}.$$

Distance between the fibers:

$$d(\xi_x, \zeta_y) = \max_{\mathcal{T}_\pm} H_\xi(T, \Delta)$$



unit is $\min(\tau, 2\pi - \tau)$

where the sign is the one of $z_\xi \doteq V_1^2 - V_2^2$,

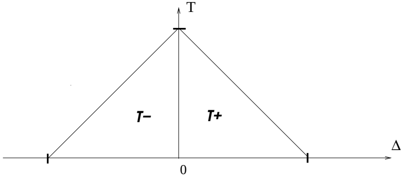
$$H_\xi(T, \Delta) \doteq T + z_\xi \Delta + W_1 \sqrt{(\tau - T)^2 - \Delta^2} + W_0 \sqrt{(2\pi - \tau - T)^2 - \Delta^2}$$

$$W_0 \doteq R \frac{|\sin(\frac{\varphi}{2})|}{|\sin \omega\pi|}, \; W_1 \doteq R \frac{|\sin(\omega\pi + \frac{\varphi}{2})|}{|\sin \omega\pi|}, \; R \doteq \sqrt{1 - z_\xi^2}.$$

- The maximum is reached for $T = 0$ or on the hypothenus.

Distance between the fibers:

$$d(\xi_x, \zeta_y) = \max_{\mathcal{T}_\pm} H_\xi(T, \Delta)$$



unit is $\min(\tau, 2\pi - \tau)$

where the sign is the one of $z_\xi \doteq V_1^2 - V_2^2$,

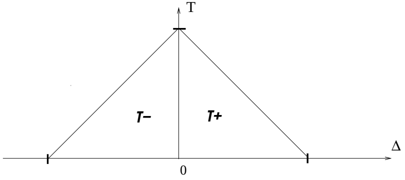
$$H_\xi(T, \Delta) \doteq T + z_\xi \Delta + W_1 \sqrt{(\tau - T)^2 - \Delta^2} + W_0 \sqrt{(2\pi - \tau - T)^2 - \Delta^2}$$

$$W_0 \doteq R \frac{|\sin(\frac{\varphi}{2})|}{|\sin \omega\pi|}, \; W_1 \doteq R \frac{|\sin(\omega\pi + \frac{\varphi}{2})|}{|\sin \omega\pi|}, \; R \doteq \sqrt{1 - z_\xi^2}.$$

- The maximum is reached for $T = 0$ or on the hypothenus.
- When $z_\xi = 0$ the maximum is reached at $\Delta = 0$.

Distance between the fibers:

$$d(\xi_x, \zeta_y) = \max_{\mathcal{T}_\pm} H_\xi(T, \Delta)$$



unit is $\min(\tau, 2\pi - \tau)$

where the sign is the one of $z_\xi \doteq V_1^2 - V_2^2$,

$$H_\xi(T, \Delta) \doteq T + z_\xi \Delta + W_1 \sqrt{(\tau - T)^2 - \Delta^2} + W_0 \sqrt{(2\pi - \tau - T)^2 - \Delta^2}$$

$$W_0 \doteq R \frac{|\sin(\frac{\varphi}{2})|}{|\sin \omega\pi|}, \; W_1 \doteq R \frac{|\sin(\omega\pi + \frac{\varphi}{2})|}{|\sin \omega\pi|}, \; R \doteq \sqrt{1 - z_\xi^2}.$$

- The maximum is reached for $T = 0$ or on the hypothenus.
- When $z_\xi = 0$ the maximum is reached at $\Delta = 0$.
- The element a that reaches the supremum has null diagonal at x, $\mathrm{Tr}(a(y)) = T$, $a_{11}(y) - a_{22}(y) = \Delta$.