Quantum Aspects of the Noncommutative Sine-Gordon Model

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Outline

1. Commutative warm-up: sine-Gordon model and a summary of well-known results.
2. Noncommutative spacetime: Moyal algebra $\mathcal{A}_\theta(\mathbb{R}^d)$ and $\star$-product.
3. Finding the model: dimensional reduction from self-dual Yang-Mills(SDYM) theory.
4. Properties of the model: classical and quantum.
5. Conclusions and outlook.

Consider the following theory for a real scalar field in $1+1$ dimensions.

$$S = \int dtdy \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + 4\alpha^2(\cos\phi - 1)\,.$$

- We use the metric $\eta_{\mu\nu} = \text{diag}(1\,, -1)$, and α has the dimensions of mass.
- The equation of motion for ϕ is

$$\partial_\mu\partial^\mu\phi = -4\alpha^2 sin\phi\,.$$

- It has kink and anti-kink solutions, which are static and given by

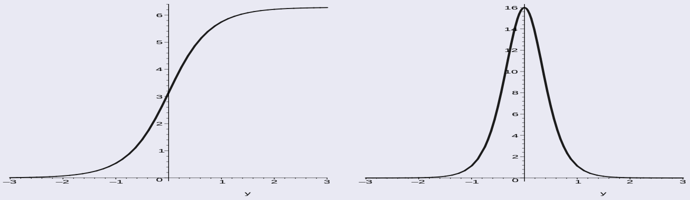
$$\phi(y) = \pm 4\arctan e^{2\alpha y}\,.$$

- Its energy density is given by

$$\epsilon = \frac{1}{2}(\partial_y \phi)^2 + 4\alpha^2(1 - \cos\phi) = \frac{16\alpha^2}{\cosh^2 2\alpha y}$$

- The kink and its energy density have the profiles

Profiles



- Its classical mass is $M_{kink} = \int dy \epsilon = 16\alpha$.
- Kink has topological charge $Q = 1$. It is disconnected from the vacuum sector with $Q = 0$.

A list of well-known properties...

1. Super-Renormalizable: It is sufficient to normal order the interactions to cancel all the divergences.

$$: 4\alpha^2(\cos\phi - 1) := 4(\alpha^2 - \delta\alpha^2)(\cos\phi - 1)$$

2. It is in fact integrable at the quantum level: Its S-matrix completely factorizes into two-particle S-matrices and obey Yang-Baxter equation. No particle production occurs!!!.
3. It has an infinite set of conserved currents.
4. It is equivalent to a fermionic theory, namely the massive Thirring model.

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- To explore the indications of the model at the quantum level, a simple analysis is to compute the corrections to M_{kink} by semi-classical means.

Quantum corrections to the kink mass

- This is done by finding the normal modes of the fluctuations around the kink solution. If ω_n are the frequencies of these modes, this implies

$$E_{kink-sector} = 16\alpha + \frac{1}{2}\hbar \sum_n \omega_n + O(\alpha^2)$$

- To find M_{kink} at this approximation, one subtracts E_{vacuum} from E_{kink} and regularizes the remaining divergences by renormalizing α. This gives

$$M_{kink} = 16\alpha - \frac{2}{\pi}\alpha + O(\alpha^2)$$

Noncommutative spacetime: Moyal algebra and $\star$-product

- Flat noncommutative spacetime is the associative algebra $\mathcal{A}_\theta(\mathbb{R}^d)$(Moyal algebra) defined via the $\star$-product:

$$(f \star g)(x) = f(x) e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} g(x)\,.$$

- The coordinate functions x_μ generate $\mathcal{A}_\theta(\mathbb{R}^d)$ and they fulfil the commutation relations

$$x_\mu \star x_\nu - x_\nu \star x_\mu =: [x_\mu\,, x_\nu]_\star = i\theta_{\mu\nu}\,.$$

- $\theta_{\mu\nu}$ is a real antisymmetric tensor of rank 2, with constant components.

We would like to have a NC sine-Gordon theory which...

Properties

- Classically Integrable: There is a linear system of equations, whose compatibility condition implies a noncommutative version of sine-Gordon field equations.
- Correct commutative limit.
- Possess kink, anti-kink solutions.
- Causal S-matrix at tree-level.

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What further properties it may have?.

- Semi-Classical behavior: spectrum of quadratic fluctuations around the vacuum and kink solutions.
- Behavior at one-loop level; quantum corrections to M_{kink}.
- SUSY extensions and their properties...

1. Consider the self-dual $U(2)$ SDYM on $\mathcal{A}_\theta(\mathbb{R}^{(2,2)})$. (We follow Lechtenfeld et. al. *Nucl.Phys.B* **705**(2005)).

$$F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\,, A_\nu]_\star$$

2. In $\mathcal{A}_\theta(\mathbb{R}^{(2,1)})$, after gauge fixing, self-duality equation becomes

$$\partial_x(\Phi^{-1}\star\partial_x\Phi) - \partial_v(\Phi^{-1}\star\partial_u\Phi) = 0\,,\ \Phi\in U(2)\,.$$

3. This is the compatibility condition for the linear system

$$(\zeta\partial_x - \partial_u)\Psi = \Phi^{-1}\star\partial_u\Phi\star\Psi\,, \quad (\zeta\partial_v - \partial_x)\Psi = \Phi^{-1}\star\partial_x\Phi\star\Psi$$

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- $\Psi(x,u,v,\zeta)$ is valued in $U(2)$ and $\zeta \in \mathbb{C}P^1$
- There is the reality condition $\Psi(\cdot,\zeta)\star\Psi^\dagger(\cdot,\bar{\zeta}) = \mathbf{1}$.
- We further have $\Psi(\cdot\,,\zeta\to 0) = \Phi^{-1}$.

- Let's assume that x-direction is commuting with t and y.
- We take the ansatz

$$\Phi(t,x,y) = V(x) \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix} V^\dagger(x)\,.$$

- $V(x) = e^{i\alpha x \sigma_1}$, $g_\pm \in U(1)$.

Compatibility equation implies

$$\begin{aligned} \partial_v(g_+^{-1} \star \partial_u g_+) + \alpha^2(g_-^{-1} \star g_+ - g_+^{-1} \star g_-) &= 0 \\ \partial_v(g_-^{-1} \star \partial_u g_-) + \alpha^2(g_+^{-1} \star g_- - g_-^{-1} \star g_+) &= 0 \end{aligned}$$

- It is possible to parameterize $g_{\pm}$ by $g_{\pm} = e_{\star}^{\mp\frac{i}{2}(\varphi\pm\rho)}$
- Commutative limit $\theta \to 0$, reproduces the standard sine-Gordon field equation:

$$\partial_u\partial_v\varphi = -4\alpha^2 sin\varphi\,, \quad \partial_u\partial_v\rho = 0\,.$$

Action

- If $\alpha = 0$, we would have had

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$$\partial_v (g_+^{-1} \star \partial_u g_+) = 0 \,, \quad \partial_v (g_-^{-1} \star \partial_u g_-) = 0 \,.$$

- These imply that the action should be consisting of WZW actions for g_+ and g_-, plus an interaction term:

$$S[g_+, g_-] = S_{WZW}[g_+] + S_{WZW}[g_-] + \alpha^2 \int dtdy \,(g_+^\dagger \star g_- + g_-^\dagger \star g_+ - 2) \,.$$

The model has the standard static kink, anti-kink solutions.

Kink, Anti-Kink

- $$\varphi_0 = \pm 4 \arctan e^{2\alpha y} \,, \quad \rho_0 = 0 \,, \quad g_0 = e^{-\frac{i}{2}\varphi_0}$$
- Multi-soliton configurations can be constructed using the linear system via the "dressing" method.
- We will study the quadratic fluctuations around the kink solution. Invoking the semi-classical reasoning, the energy spectrum for the kink particle will be

$$E_{kink-sector} = 16\alpha + \frac{1}{2}\sum_n (\omega_n + \nu_n) + O(\alpha^2)$$

where ω_n and ν_n are the frequencies for the normal modes.

- Let's split the fields g_+ , g_- by setting

$$g_+ = g_0 e^{-i(\eta+\xi)}\,, \qquad g_- = e^{i(\eta-\xi)} g_0^{-1}\,,$$

- $\eta\,,\xi$ are fluctuations in the static background g_0.
- We expand $S[g_+, g_-]$ up to cubic order in η and ξ.

$$S[g_+, g_-] = S[g_0] - \int dtdy\,(\partial_\mu\eta)^2 + (\partial_\mu\xi)^2 + \text{interaction terms}$$

1. First, we find the field equations for η and ξ and expand them to second order in θ.
2. Next, we expand the fluctuations in modes by assuming

$$\eta(t,y) = \sum_n e^{i\omega_n t}\psi_n(y)\,, \quad \xi(t,y) = \sum_n e^{i\nu_n t}\chi_n(y)\,.$$

Eigenmodes fulfil the Schrödinger-type equations:

Equations, ($z := 2\alpha y$)

$$\left[-\partial_z^2 + V_0(z) + \theta V_1(z) + \theta^2 V_2(z) \right] \tilde{\psi}_n(z) = \frac{\omega_n^2}{4\alpha^2} \tilde{\psi}_n(z) \, ,$$

$$\left[-\partial_z^2 + \theta W_1(z) + \theta^2 W_2(z) \right] \tilde{\chi}_n(z) = \frac{\nu_n^2}{4\alpha^2} \tilde{\chi}_n(z) \, .$$

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Potentials

$$V_0 = (2\tanh^2 z - 1) \,, \; V_1 = -\omega_n^2 \frac{\sinh z}{\cosh^2 z}$$

$$V_2 = -\omega_n^2 \alpha^2 \left(\frac{2}{\cosh^4 z} - \frac{\sinh^2 z}{\cosh^4 z} \right)$$

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Potentials

$$W_1(z) = -\nu_n^2 \frac{\sinh z}{\cosh^2 z}$$

$$W_2(z) = \nu_n^2 \alpha^2 \frac{\sinh^2 z}{\cosh^4 z}$$

We consider θ-dependent potentials as perturbations.

1. For $\theta = 0$, the spectrum is exactly known. It consists of a zero mode followed by a continuum of states.

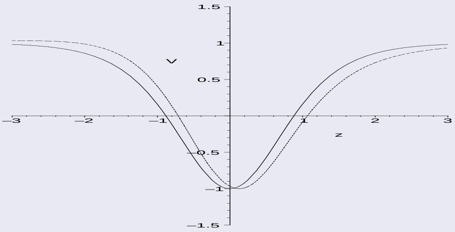
$$\psi_0(z) = \partial_z \varphi_0 = -\frac{2}{\cosh z}, \quad \psi_q(z) = e^{iqz}(\tanh z - iq).$$

2. $\psi_0(z) = -\frac{2}{\cosh z}$ is static, and remains a zero-mode to all orders in θ.
3. At order θ: V_1 and W_1 are odd under parity, so first order perturbations in θ give no corrections to the normal frequencies.

At order θ^2:

- Corrections to normal frequencies due to V_2 and W_2 via first order perturbation theory in θ^2 also vanish.
- It does not seem possible to obtain analytic results for V_1 and W_1 at second order in perturbation theory. Qualitatively, it seems unlikely that they change the spectrum considerably:

Potential Profiles



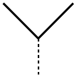
Propagators

- $$\text{———} \quad \equiv \langle\varphi\varphi\rangle = \frac{2}{k^2 + 4\alpha^2}\,, \qquad \cdots\cdots\cdots \quad \equiv \langle\rho\rho\rangle = \frac{2}{k^2}$$

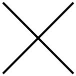
Vertices

- The vertices at quartic order in the fields φ and ρ

Feynman rules for these vertices read

- 

$$= -\frac{1}{2^2}(k_1 \wedge k_2) \sin\left(\theta \frac{k_1 \wedge k_2}{2}\right) e^{-\frac{i}{2}\theta(k_1 \wedge k_2 + k_2 \wedge k_3)}$$

- 

$$= \frac{1}{12}\alpha^2 e^{\left(-\frac{i}{2}\theta \sum_{i<j}^{n} k_i \wedge k_j\right)} - \frac{i}{2^2 \cdot 4!} k_1 \cdot (k_3 - k_2)$$
$$\times \sin\left(\theta \frac{k_2 \wedge k_3}{2}\right) e^{-\frac{i}{2}\theta(k_1 \wedge k_2 + k_1 \wedge k_3 + k_1 \wedge k_4 + k_2 \wedge k_4 + k_3 \wedge k_4)}$$

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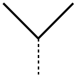
- 
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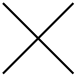
- 
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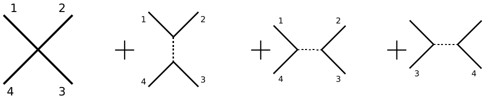
- $a\wedge b = a_t b_y - a_y b_t$

Scattering amplitudes

It was shown by Lechtenfeld et. al.*Nucl.Phys.B***705**(2005)) that this model do not exhibit any acausal behavior at tree level.

- $A_{\varphi\varphi\to\varphi\varphi} =$

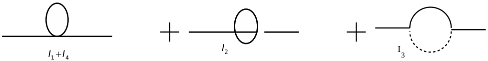


$= 2i\alpha^2$

- All other amplitudes, $A_{\rho\rho\to\rho\rho}$, $A_{\varphi\rho\to\varphi\rho}$, $A_{\varphi\varphi\to\rho\rho}$ and $A_{\rho\rho\to\varphi\varphi}$ vanish.
- Thus the model has no acausal effects.
- Amplitudes for $\varphi\varphi \to \varphi\varphi\varphi\varphi$ and $\varphi\varphi\varphi \to \varphi\varphi\varphi$ also vanish. This is in agreement with the commutative sine-Gordon model.

One-loop two-point functions in vacuum sector...

- Two-point function for φ is $I_\varphi(P^2)$

$I_\varphi(P^2) =$ 

- Non-planar diagram $I_2(P^2)$ leads to UV/IR mixing. We observe this from

$$I_2(P^2) = \frac{-\alpha^2}{6\pi} \log\left[\alpha^2\theta^2 P^2 + \frac{4\alpha^2}{\Lambda^2}\right] + \text{subleading terms}\,,$$

- $I_3(P^2)$ and $I_4(P^2)$ are present purely due to the noncommutativity, they vanish as $\theta \to 0$. There is no UV/IR mixing due to $I_1(P^2)$, $I_3(P^2)$ and $I_4(P^2)$.
- $I_\rho(P^2)$ is present also purely due to the noncommutativity, but it does not lead to any UV/IR mixing.

Renormalization in the Euclidean signature.

- For $P \neq 0$, $\theta \neq 0$, the leading terms for $I_\varphi(P^2)$ reads

$$I_\varphi(P^2) \approx \left[\frac{-\alpha^2}{3\pi} + \frac{P^2}{2^6\pi}\right] \log \frac{4\alpha^2}{\Lambda^2} + \text{finite terms} + \text{subleading terms}$$

- Mass and field strength counter terms are found using standard renormalization methods.
- There is only field strength renormalization for the field ρ.
- **Remark1:** When $\theta \to 0$ the standard answer for the commutative sine-Gordon model is recovered.
- **Remark2:** $I(P^2)$ leads to unitarity violation, when it is analytically continued to the Minkowski space.

SUSY extensions

A natural $N = 1$ SUSY extension of the action is

$$S = S_{SWZW}[G_+] + S_{SWZW}[G_-] - 2\alpha \int dtdyd^2\theta G_+^{-\frac{1}{2}} \star G_-^{\frac{1}{2}} + G_-^{-\frac{1}{2}} \star G_+^{\frac{1}{2}}$$

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$$S_{SWZW}(G) = \frac{1}{2} \int dtdyd^2\theta \bar{D}G^{-1} \star DG + \frac{1}{2} \int dtdyd^2\theta d\lambda G^{-1} \partial_\lambda G \star \bar{D}G^{-1} \star \gamma_5 DG .$$

- D and $\bar{D}$ are standard SUSY covariant derivatives.
- Standard SUSY kink is a solution of the field equations.
- Classical integrability of the field equations are under investigation. It seems that there indeed exits a linear system for this model, its details are being worked out.
- It will certainly be useful to see if SUSY helps in regularizing the divergences of the bosonic theory.

Summarizing...

1. We have studied the quantum aspects of sine-Gordon model in noncommutative spacetime. Our aim has been to infer to what extent the classical integrability is useful in this respect.
   - We have presented a perturbative treatment of noncommutativity to study the spectrum of fluctuations around the kink. This implied that the latter is in good agreement with that of the ordinary sine-Gordon model.
2. Two-point functions at one-loop level show UV/IR mixing due to interactions coupled via α^2, but it appears that there are non-planar diagrams which do not lead to UV/IR mixing effects.
   - In Euclidean signature, mass and field strength renormalizations are obtained for non-exceptional momenta. However, in Minkowski signature there is still unitarity violation.
   - Although, the usual vacuum subtraction can be performed it is not clear, how to regularize the divergences of the theory in Minkowski space.

3. It maybe be helpful to study the quantum effects in the $2+1$-dimensional Ward-model to gain more insights on the structure of the present class of models.
4. It will certainly be useful to study the SUSY generalizations of this model and see if it helps in regularizing the divergences of the bosonic theory. Investigations in this direction are already underway.