Cohomological analysis of the Epstein–Glasser renormalization

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Introduction

$$\int_{\mathbb{R}^{Dn}} \prod_{1 \leqslant j < k \leqslant n} G_{jk}(\mathsf{x}_j - \mathsf{x}_k) \prod_{m=1}^{n} d^D \mathsf{x}_m$$

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$$\mathcal{R} : \mathcal{C}^{\infty}\Big(F_n\big(\mathbb{R}^D\big)\Big) \to \mathscr{D}'\big(\mathbb{R}^{Dn}\big)\,,$$

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Then: $\mathcal{Q} := \mathcal{R} - \mathcal{R}' : \mathcal{C}^{\infty}\Big(F_n\big(\mathbb{R}^D\big)\Big) \to \mathscr{D}'\big(\widehat{\boldsymbol{\Delta}}_n\big)$

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Conversely, $\quad \mathcal{R}' := \mathcal{R} + \mathcal{Q} \quad$ for $\quad \mathcal{Q} : \mathcal{C}^\infty\Big(F_n\big(\mathbb{R}^D\big)\Big) \to \mathscr{D}'\big(\widehat{\Delta}_n\big)$

$$\Rightarrow \qquad \mathcal{R}'(u)\Big|_{F_n(\mathbb{R}^D)} = u.$$

$$c\Big[A\Big] := A \circ \mathcal{R} - \mathcal{R} \circ A \;:\; \mathcal{C}^{\infty}\Big(F_n\Big(\mathbb{R}^D\Big)\Big) \longrightarrow \mathscr{D}'\Big(\widehat{\boldsymbol{\Delta}}_n\Big)$$

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$c\left[A\right]$ is a Hochschild 1 – cocycle :

$$A_1 \circ c\left[A_2\right] - c\left[A_1 \circ A_2\right] + c\left[A_1\right] \circ A_2 = 0\,.$$

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for $\quad c'\left[A\right] := A \circ \mathcal{R}' - \mathcal{R}' \circ A$

$c\left[A\right] - c'\left[A\right]$ is a Hochschild coboundary :

$$c\left[A\right] - c'\left[A\right] = A \circ \mathcal{Q} - \mathcal{Q} \circ A \, .$$

Recursive renormalization

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$$\text{if} \quad (\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3) \notin \Delta_3$$

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$$G_{12}(x_1 - x_2)\; G_{13}(x_1 - x_3)\; G_{14}(x_1 - x_4)\; G_{24}(x_2 - x_4)\; G_{34}(x_3 - x_4)$$

There exist $\mathcal{R}_n : \mathcal{C}^\infty\Big(F_n\big(\mathbb{R}^D\big)\Big) \to \mathscr{D}'\Big(\mathbb{R}^{Dn}\Big)$ with the properties

$$\mathcal{R}_m\Big(\mathcal{R}_{n-m}(u)\Big) = \mathcal{R}_n\Big(u\Big|_{F_n(\mathbb{R}^D)}\Big)$$

$$\text{if} \quad u \in \mathcal{C}^\infty\Big(F_m\big(\mathbb{R}^D\big) \times F_{n-m}\big(\mathbb{R}^D\big)\Big)$$

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$$\mathcal{R}_n\big(fu\big) = f\,\mathcal{R}_n(u),$$

$$\text{if} \quad u \in \mathcal{C}^\infty\Big(F_n\big(\mathbb{R}^D\big)\Big) \quad \text{and} \quad f \in \mathcal{C}^\infty\Big(\mathbb{R}^{Dn}\Big).$$

$$\mathcal{R}_n\Big[u(\mathsf{x}_1-\mathsf{x}_2,\ldots,\mathsf{x}_1-\mathsf{x}_n)\Big] = R_n\Big[u(\mathsf{y}_1,\ldots,\mathsf{y}_{n-1})\Big]$$

$$(\mathsf{x}_1,\ldots,\mathsf{x}_n) \mapsto (\mathsf{x}_1-\mathsf{x}_2,\ldots,\mathsf{x}_1-\mathsf{x}_n) : \begin{cases} F_n\Big(\mathbb{R}^D\Big) & \longrightarrow & F_{n-1}\Big(\mathbb{R}^D\Big\backslash\{0\}\Big) \\ \widehat{\boldsymbol{\Delta}}_n & \longrightarrow & \widehat{\boldsymbol{\Delta}}_{n-1} \\ \boldsymbol{\Delta}_n & \longrightarrow & 0 \end{cases}$$

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$$R_n = \mathcal{P}_{D(n-1)} \circ R'_n$$

$$R'_n : \mathcal{C}^\infty\Big(F_{n-1}\big(\mathbb{R}^D\backslash\{0\}\big)\Big) \to \mathscr{D}'\Big(\mathbb{R}^{D(n-1)}\backslash\{0\}\Big)$$

$$\mathcal{P}_N : \mathscr{D}'\big(\mathbb{R}^N\backslash\{0\}\big) \to \mathscr{D}'\big(\mathbb{R}^N\big)$$

$$R_2 = \mathcal{P}_D$$

There exist $\mathcal{P}_N : \mathscr{D}'\big(\mathbb{R}^N \setminus \{0\}\big) \to \mathscr{D}'\big(\mathbb{R}^N\big)$ with the properties

$$\mathcal{P}_N(u)\Big|_{\mathbb{R}^N \setminus \{0\}} = u\,,$$

$$\text{sc.d.}\, \mathcal{P}_N(u) \leqslant \text{sc.d.}\, u,$$

$$\mathcal{P}_N\big(fu\big) = f\,\mathcal{P}_N(u),$$

if $u \in \mathscr{D}'\big(\mathbb{R}^N \setminus \{0\}\big)$ and $f \in \mathcal{C}^\infty\big(\mathbb{R}^N\big)$.

$$\mathcal{P}_N = \mathcal{P}'_N + \mathcal{Q}$$

$$\mathcal{Q} : \mathscr{D}'\Big(\mathbb{R}^N \setminus \{0\}\Big) \to \mathscr{D}'\Big[0\Big]$$

$$c_\xi := x^\xi \circ \mathcal{P}'_N - \mathcal{P}'_N \circ x^\xi : \mathscr{D}'\Big(\mathbb{R}^N \setminus \{0\}\Big) \to \mathscr{D}'\Big[0\Big]$$

$$c_\xi = x^\xi \circ \mathcal{Q} - \mathcal{Q} \circ x^\xi$$

$$\mathcal{Q} = \sum_{\mathsf{r} \in \mathbb{N}_0^N} \frac{1}{\mathsf{r}!}\, \delta^{(\mathsf{r})}\Big(\mathbf{x}\Big)\, Q_{\mathsf{r}}\,, \quad Q_{\mathsf{r}} : \mathscr{D}'_t\Big(\Big(\mathbb{R}^N \setminus \{0\}\Big) \to \mathbb{R}$$

$$c_\xi = \sum_{\mathsf{r} \in \mathbb{N}_0^N} \frac{1}{\mathsf{r}!}\, \delta^{(\mathsf{r})}\Big(\mathbf{x}\Big)\, C_{\xi,\mathsf{r}}\,, \quad C_{\xi,\mathsf{r}} : \mathscr{D}'_t\Big(\mathbb{R}^N \setminus \{0\}\Big) \to \mathbb{R}$$

$$Q_{\mathsf{r}+\mathsf{e}_\xi}\Big[u\Big] = -\,C_{\xi,\mathsf{r}}\Big[u\Big] - Q_{\mathsf{r}}\Big[x^\xi\, u\Big]$$

Cohomology of renormalization maps

$$R_n = \mathcal{P}_{D(n-1)} \circ R'_n$$

$$\mathcal{C}^\infty\Big(F_{n-1}\big(\mathbb{R}^D \backslash \{0\}\big)\Big) \xrightarrow{R'_n} \mathscr{D}'\Big(\mathbb{R}^{D(n-1)} \backslash \{0\}\Big) \xrightarrow{\mathcal{P}_{D(n-1)}} \mathscr{D}'\big(\mathbb{R}^N\big)$$

$$\omega_{n;\,k,\,\mu} := R_n \circ \partial_{x_k^\mu} - \partial_{x_k^\mu} \circ R_n =: \big[R_n, \partial_{x_k^\mu}\big]$$

$$\omega_{n;\,k,\,\mu} : \mathcal{C}^\infty\Big(F_{n-1}\big(\mathbb{R}^D \backslash \{0\}\big)\Big) \to \mathscr{D}'\big(\widehat{\Delta}_{n-1}\big)$$

$$\omega_{n;\,k,\,\mu} = \gamma_{n;\,k,\,\mu} + \omega'_{n;\,k,\,\mu}$$

$$\gamma_{n;\,k,\,\mu} := \big[\partial_{x_k^\mu}, \mathcal{P}_{D(n-1)}\big] \circ R'_n$$

$$\omega'_{n;\,k,\,\mu} := \mathcal{P}_{Dn} \circ \big[\partial_{x_k^\mu}, R'_n\big]$$

$$\gamma_{n;\,k,\,\mu} : \mathcal{C}^\infty\Big(F_{n-1}\big(\mathbb{R}^D \backslash \{0\}\big)\Big) \to \mathscr{D}'\big[0\big]$$

$$\left[\partial_{x_j^\nu}, \gamma_{2;\,k,\,\mu}\right] - \left[\partial_{x_k^\mu}, \gamma_{2;\,j,\,\nu}\right] = 0$$

$$\left[\partial_{x_j^\nu}, \gamma_{n;\,k,\,\mu}\right] - \left[\partial_{x_k^\mu}, \gamma_{n;\,j,\,\nu}\right]$$

$$= -\left[\partial_{x_k^\mu}, \mathcal{P}_{D(n-1)}\right] \circ \left[\partial_{x_j^\nu}, J_n'\right] + \left[\partial_{x_j^\nu}, \mathcal{P}_{D(n-1)}\right] \circ \left[\partial_{x_k^\mu}, J_n'\right] \quad (n > 2)$$

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$$d\boldsymbol{\gamma}_2 = 0$$

$$d\boldsymbol{\gamma}_n = \mathcal{F}\left[\gamma_1, \ldots, \gamma_{n-1}\right]$$

$$\phi : \mathcal{C}^{\infty}\Big(F_{n-1}\big(\mathbb{R}^{D} \setminus \{0\}\big)\Big) \to \mathscr{D}'[0]$$

$$x_k^{\mu} \circ \phi - \phi \circ x_k^{\mu} = 0$$

$$\phi : \mathcal{C}^{\infty}\Big(F_{n-1}\big(\mathbb{R}^{D} \setminus \{0\}\big)\Big) \to \mathscr{D}'\big[0\big]$$

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$$\phi = \sum_{\mathrm{r} \in \mathbb{N}_0^N} \frac{1}{\mathrm{r}!}\, \delta^{(\mathrm{r})}\big(\mathbf{x}\big)\, \Phi_{\mathrm{r}}\,, \qquad \Phi_{\mathrm{r}} : \mathcal{C}^{\infty}\Big(F_{n-1}\big(\mathbb{R}^{D} \setminus \{0\}\big)\Big) \to \mathbb{R}\,.$$

$$\phi \mapsto \Phi_0 \quad \text{is injective}$$

$$\Phi_{\mathrm{r}} = (-1)^{|\mathrm{r}|}\, \Phi_0 \circ \mathbf{x}^{\mathrm{r}}$$

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$\phi \mapsto \Phi_0$ is injective

$$\Phi_{\mathsf{r}} = (-1)^{|\mathsf{r}|}\, \Phi_0 \circ \mathbf{x}^{\mathsf{r}}$$

if $\phi \mapsto \Phi_0$ then $\big[\partial_{x^{\xi}}, \phi\big] \mapsto -\Phi_0 \circ \partial_{x^{\xi}}\,.$

$$H^{1}\Big(\mathcal{C}^{\infty}\big(F_{n-1}\big(\mathbb{R}^{D}\setminus\{0\}\big)\big)^{\circ}\Big)$$

$$H^{D(n-1)-1}\Big(F_{n-1}\big(\mathbb{R}^{D}\setminus\{0\}\big)\Big)$$

$$H^1\Big(\mathcal{C}^\infty\big(F_{n-1}(\mathbb{R}^D\setminus\{0\})\big)^\circ\Big)$$

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$$H^{D(n-1)-1}\Big(F_{n-1}\big(\mathbb{R}^D\setminus\{0\}\big)\Big) = \{0\} \qquad \text{for} \qquad n > 2$$

$$\psi : \mathbb{S}^{D-1} \to F_2\big(\mathbb{R}^D\big) : \mathsf{x} \mapsto \big(\mathsf{x}, -\mathsf{x}\big)$$

$$\pi^n_{j_1,\dots,j_m} : F_n\big(\mathbb{R}^D\big) \to F_m\big(\mathbb{R}^D\big) : \big(\mathsf{x}_1,\dots,\mathsf{x}_n\big) \mapsto \big(\mathsf{x}_{j_1},\dots,\mathsf{x}_{j_m}\big)$$

$$(1 \leqslant j_1 < \cdots < j_m \leqslant n)$$

$$\alpha_{j,k} := \big(\pi^n_{j,k}\big)^* \alpha \,, \quad \alpha \in \Omega^{D-1}\big(\mathbb{S}^{D-1}\big)$$

The spaces $H^r\Big(F_n\big(\mathbb{R}^D \setminus Q\big)\Big)$ are finite dimensional and the only nonzero ones are for $r = s\,(D-1)$ with $s = 1,\dots,n-1$.

The algebra $H^*\Big(F_n\big(\mathbb{R}^D\big)\Big)$ is a free algebra with generators $[\alpha_{j,k}]$ for $1 \leqslant j < k \leqslant n$ and relations

$$[\alpha_{j,k}]^2 = 0 \quad (j < k)\,,$$
$$[\alpha_{j,\ell}][\alpha_{k,\ell}] = [\alpha_{j,k}][\alpha_{k,\ell}] - [\alpha_{j,k}][\alpha_{j,\ell}] \quad (j < k < \ell)\,.$$