Simulating $U(1)$ Gauge Theory on Non-Commutative Spaces

I. **NC $U(1)$ gauge theory**

II. **Wilson loops in $d = 2$: area-preserving diffeomorphisms**

III. **NC QED_4 : the fate of the NC photon**

W.B., A. Bigarini, J. Nishimura, Y. Susaki, A. Torrielli and J. Volkholz

I. NC $U(1)$ gauge theory

- **NC Euclidean plane**

NC space coordinates in $d = 2$:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\Theta_{\mu\nu} = i\theta\epsilon_{\mu\nu} \qquad \begin{array}{ccl} \hat{x}_\mu & : & \text{Hermitian operators} \\ \theta & : & \text{NC parameter, const.} \end{array}$$

imply spatial uncertainty

$$\Delta x_1 \, \Delta x_2 \sim \theta$$

(cf. event horizon of a strong gravitation centre), and non-locality.

UV/IR mixing of divergences
$\longrightarrow$ perturbation theory beyond one loop is mysterious.

- **Lattice structure**

Non-perturbative approach :

Imposing the operator identity

$$\exp\left(i\frac{2\pi}{a}\hat{x}_\mu\right) = \hat{\mathbb{1}}$$

yields a (fuzzy) lattice structure.

Periodicity over the Brillouin zone $\longrightarrow$ lattice is also spatially periodic:

$$\frac{1}{2a}\theta\, p_\mu \in \mathbb{Z}$$

Periodic $N \times N$ lattice

$$\Rightarrow \qquad \underline{\theta = \frac{1}{\pi} N a^2}$$

Double Scaling Limit

$$\left.\begin{array}{r} a \to 0 \\ N \to \infty \end{array}\right\} \quad \text{at} \quad N a^2 = const.$$

leads to a **continuous NC plane of infinite extent**.

Simultaneous UV and IR limit.

Return to ordinary coordinates x_μ, if all fields are multiplied by $\star$-products:

$$\phi(x) \star \psi(x) := \phi(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \Theta_{\mu\nu} \overrightarrow{\partial}_\nu\right) \psi(x)$$

e.g. $[x_\mu, x_\nu]_\star = i\Theta_{\mu\nu}$

$U(1)$ Gauge Theory

$$S[A] = \frac{1}{4} \int d^2x \; F_{\mu\nu} \star F_{\mu\nu} \;, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig(A_\mu \star A_\nu - A_\nu \star A_\mu)$$

S is $\star$-gauge invariant (self-interaction !)

Cannot be simulated in this form on the lattice ($\star$-unitary link variables)

Way out: equivalence to a matrix model on one point

Twisted Eguchi-Kawai Model (González-Arroyo/Okawa, '83)

$$S_{\mathrm{TEK}}[U] = -N\beta \sum_{\mu\neq\nu} Z_{\mu\nu} \mathrm{Tr}[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger]$$

U_μ: unitary $N \times N$ matrices , $\beta \equiv 1/g^2$
Twist: $Z_{21} = Z_{12}^* = \exp(2\pi i n/N)$, here: $n = \frac{N+1}{2}$ $\rightarrow$ N odd

Morita equivalence (identical algebras) Aoki et al., '99

$$\mathrm{TEK}_{N\to\infty} \Leftrightarrow \text{ NC } U(1) \text{ gauge theory on infinite lattice}$$

Refinement (Ambjørn et al., '00) :

$$\mathrm{TEK}_{N \text{ finite}} \Leftrightarrow \text{ NC } U(1) \text{ gauge theory on } N \times N \text{ lattice}$$

Mapping back the matrix model term (Ishibashi et al., Gross et al., '00)

$$W_{\mu\nu}(I \times J) := \frac{1}{N} Z_{\mu\nu}^{I \cdot J} \, \mathrm{Tr}[U_\mu^I U_\nu^J U_\mu^{\dagger\, I} U_\nu^{\dagger\, J}]$$

to the lattice defines the **NC Wilson loop.**

Note that $W_{\mu\nu} \in \mathbb{C}$, but the action is real
(both orientations summed over, $W_{\mu\nu} = W_{\nu\mu}^*$)

$\Rightarrow$ In this form, Monte Carlo simulations are possible !

(W.B./Hofheinz/Nishimura, '02)

Scale : $\beta = 1/a^2$ from Gross-Witten area law in the planar limit
$\rightarrow$ fix N/β for Double Scaling

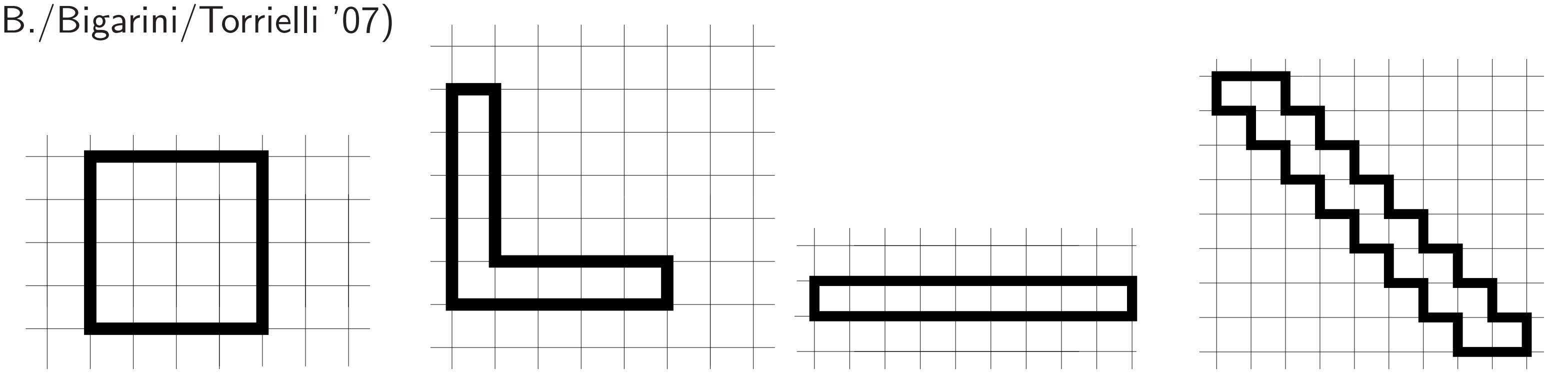
II. Wilson loops in $d = 2$: area-preserving diffeomorphisms (APDs)

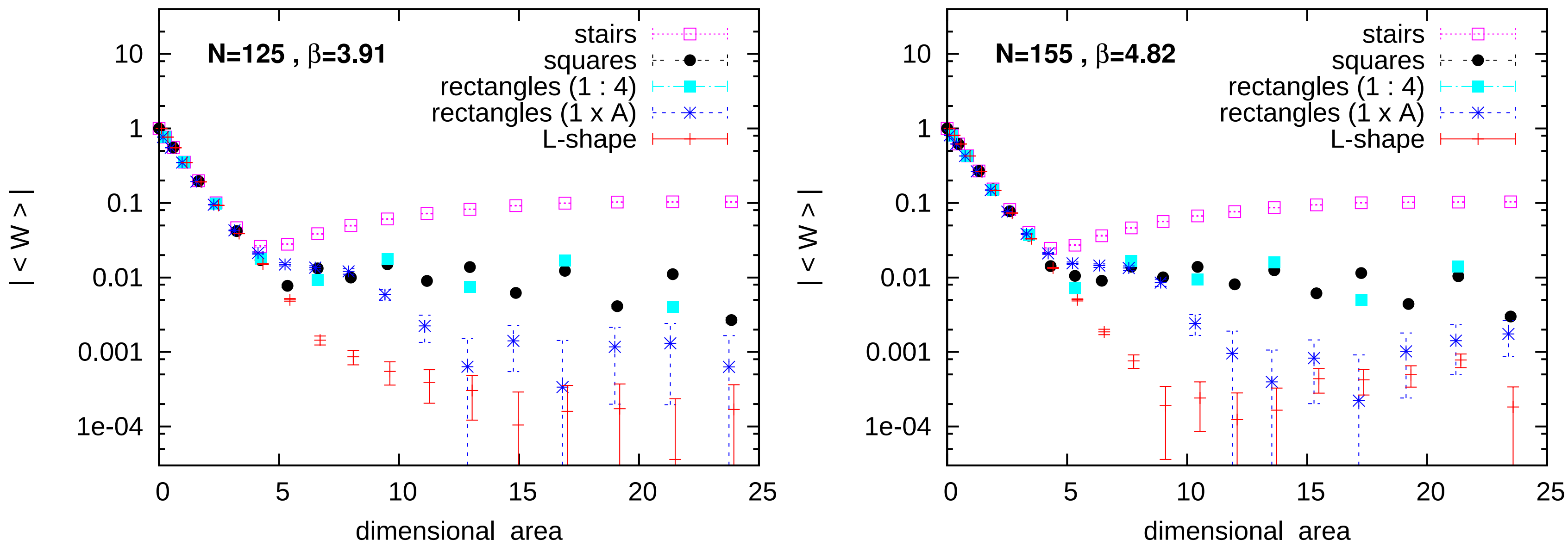
Pure YM theories on a *commutative* plane: soluble thanks to APD invariance; $\langle \text{Wilson loop} \rangle$ only depends on (oriented) area.

NC $U(n)$: Perturbation theory to $O(g^4\theta^{-2})$ reveals sym. breaking down to $SL(2, R)$. (Ambjørn/Dubin/Makeenko '04, Bassetto/De Pol/Torrielli/Vian '05)

Non-perturbative test for NC $U(1)$ with squares, L-shapes, rectangles, stairs

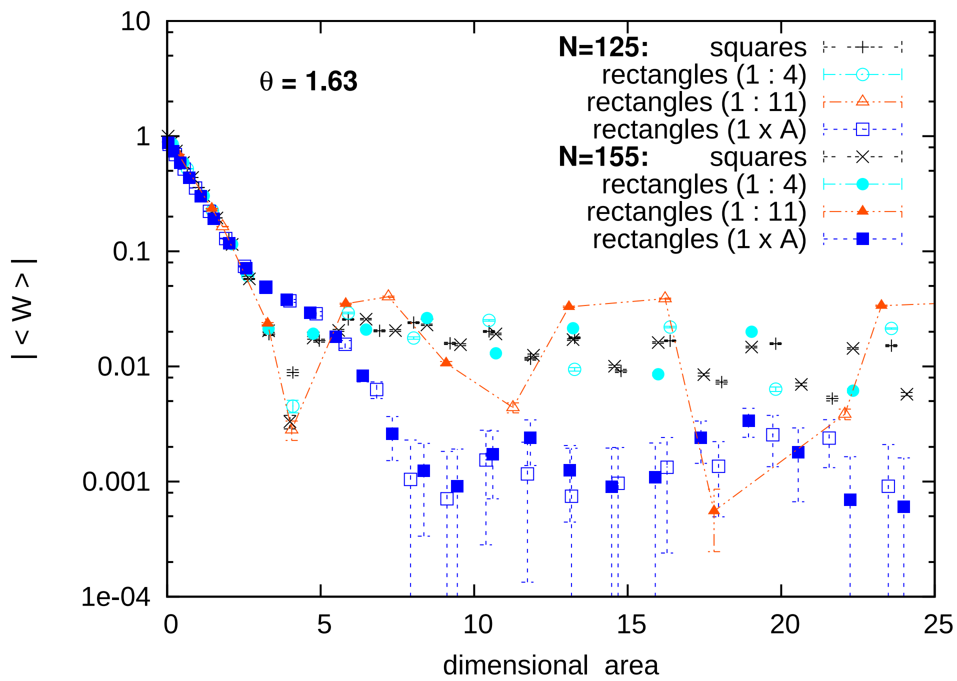
(W.B./Bigarini/Torrielli '07)





$| < W > |$ for various shapes surrounding the same area,
at $\theta \equiv \frac{1}{\pi} N a^2 = 2.63$.

Results for different volumes $(Na)^2$ and different a agree.
Area law at small (dimensional) area, but deviations beyond
$\longrightarrow$ shape dependence persists in the Double Scaling Limit.



Focus on the rectangles to check sym. under the APD subgroup SL(2,R) .

Here we fix $\theta = 1.63$ at different volumes and lattice spacings:
we see again DSL convergence with (minor) sym. breaking.

$\Rightarrow$ On the non-pert. level, the **APD sym. breaks completely.**

Analytic solution hard to find. Simulations are crucial, as in 4d YM theory.

III. NC QED$_4$: the fate of the NC photon

We consider again a NC plane $[\hat{x}_1, \hat{x}_2] = i\theta = const.$, plus x_3, t : commutative.

NC photon: <u>Θ-deformed dispersion relation ?</u>

1-loop calculations suggest the form (Matusis/Susskind/Thoumbas, '00)

$$E^2 = \vec{p}^{\,2} + \frac{C}{(p\Theta)^2}$$

Test with data from cosmic photons ! General ansatz:

$$E = |\vec{p}|\left(1 + \frac{E}{M}\right) \qquad (\, M \ : \ ''\text{quantum gravity foam}''\,)$$

E.g. different time of flight for 35 Gamma Ray Bursts

$\Rightarrow$ $M > 0.001\, M_{\text{Planck}}$ (Ellis/Mavromatos/Nanopoulos/Sakharov/Sarkisyan , '04)

Bounds for θ in Nature ? (Amelino-Camelia et al. '98)

But: 1-loop perturbation theory : IR singularity is negative $(C < 0)$

(Landsteiner/Lopez/Tytgat, '00, . . .)

IR instability, ill defined, to be cured by SUSY . . .

- **NC QED$_4$ revisited non-perturbatively** (W.B./Nishimura/Susaki/Volkholz, '06)

- comm. plane $\rightarrow$ $L \times L$ lattice
- NC plane $\rightarrow$ matrix model (TEK) $(N \approx L)$

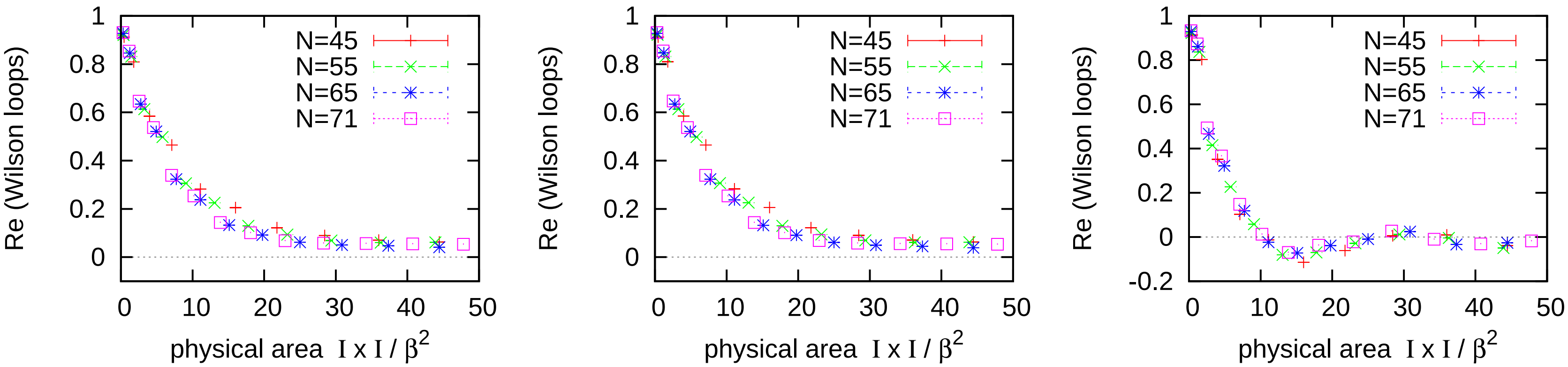
First goal: search for physical scale $\rightarrow$ identify a Double Scaling Limit.

Successful ansatz:

$$a \propto 1/\beta \qquad \rightarrow \qquad \underline{\theta \propto N/\beta^2}$$

(different from NC QED$_2$, fine-tuned scaling at each N differs slightly)

Double Scaling for the Wilson loops : (here $N/\beta^2 \equiv 20$)



commutative plane — mixed plane — NC plane

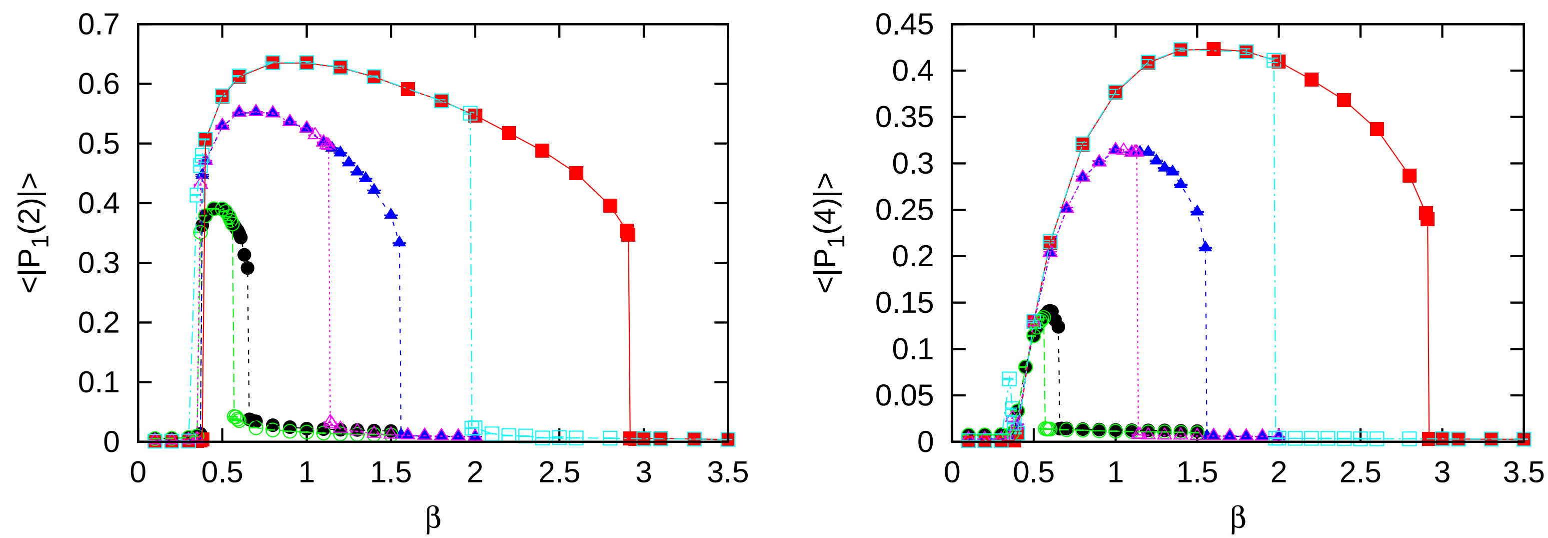
comm. and mixed planes: $\langle W \rangle \in \mathbb{R}$ due to sym. in signs of x_3 and t

Order parameter for translation sym. in NC plane : open Wilson line

$\star$-gauge invariant, carries momentum p (Ishibashi/Iso/Kawai/Kitazawa '00)

$$|P_\mu(n)| := \frac{1}{N}|\mathrm{Tr}[U_\mu^n]|$$

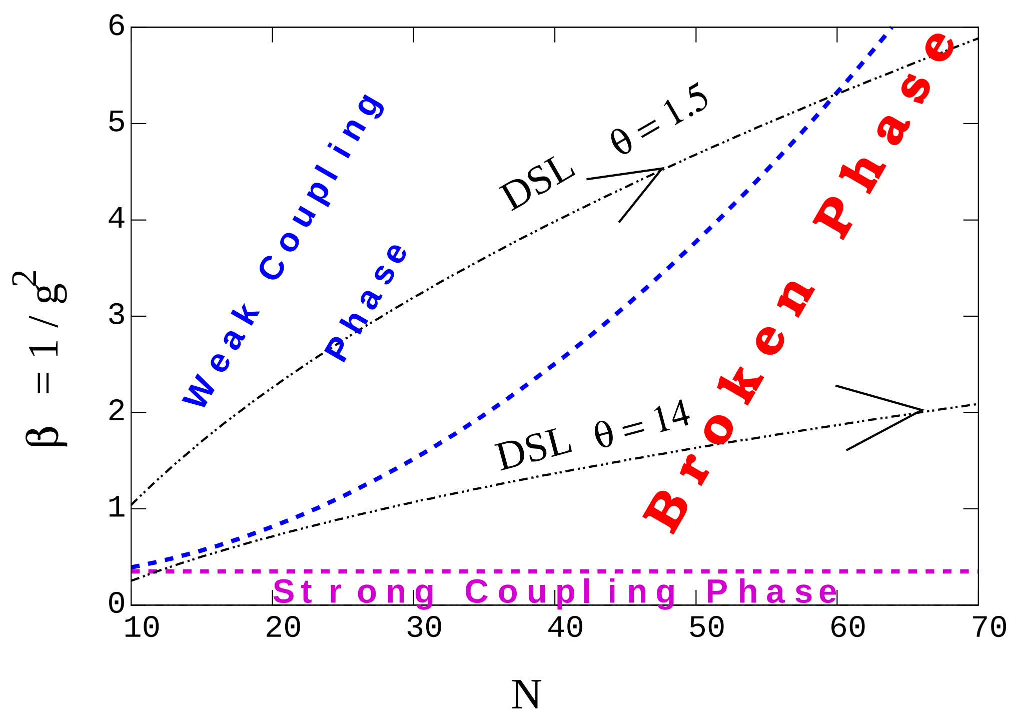
Wilson line of lenght $\tilde{p}_\mu \equiv \Theta_{\mu\nu} p_\nu = n a \hat{\mu}$



$|P_1|$ for $N = 15,\ 25,\ 35$ Coupling: **strong** | **moderate** | | **weak**

$\beta \approx 0.35$ hysteresis

Phase diagram : Weak ↔ Moderate ↔ Strong coupling (cf. Ishikawa/Okawa, Teper/Vairinhos)



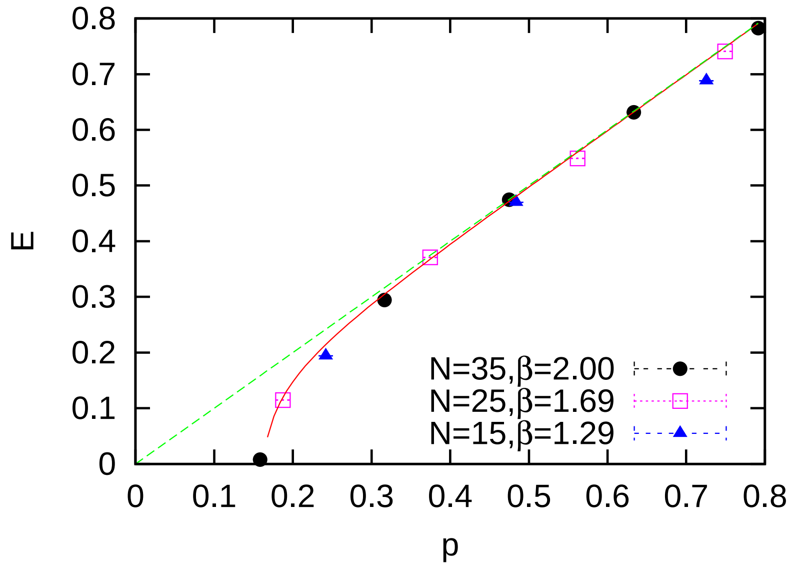
Double Scaling Limit $\beta \sim \sqrt{N}$ always leads to the **broken** phase $(Na^2 \simeq N/\beta^2)$.

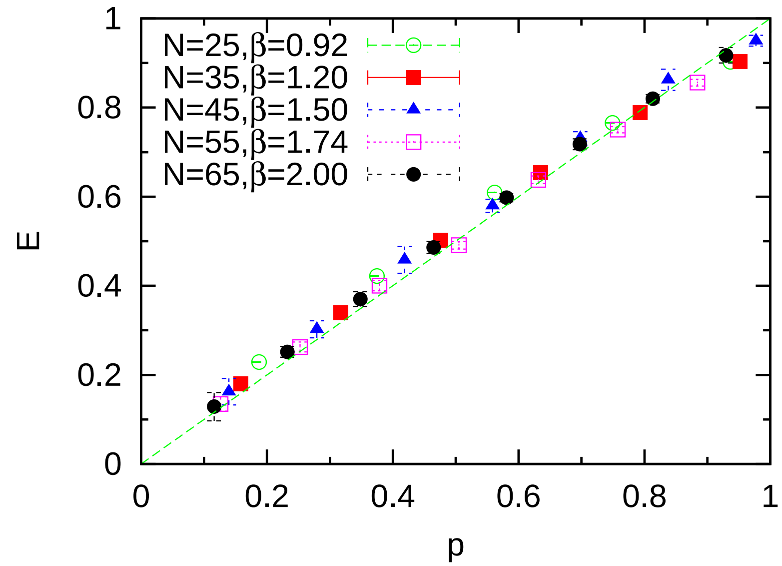
In this limit, we found stability of all observables that could be measured well.

Broken phase could describe a stable cont. limit for the NC photon.

Dispersion Relation

determined from exp. decay in the comm. plane $E(p = p_3)|_{p_1=p_2=0}$





sym. phase
consistent with
neg. IR divergence
("tachyonic" behaviour)

broken phase
IR stable
Nambu-Goldstone boson
(SSB of transl. sym.)

V. Conclusions

We studied QED_2, and QED_4 on spaces with a NC plane.

Discretised NC plane can be mapped onto a **TEK matrix model**;
enables MC simulations (heat-bath after linearisation through auxiliary matrix field)

A Double Scaling Limit to a continuous, infinite NC space converges
$\longrightarrow$ non-pert. renormalisable.

- **QED_2 :**

small area : Wilson loops obey area law for all shapes
large area : complex phase for many shapes $=$ (area / θ) ,
corresponds to AB effect with $B = 1/\theta$ (cf. W.B./Hofheinz/Nishimura '02)

At large area the **APD sym. is broken, including the SL(2,R) subgroup.**
$\Rightarrow$ Unlike 2d YM theory: not analytically soluble with commutative techniques
but rich structure, can be explored numerically.

- **QED$_4$:**

Double Scaling Limit identified by matching of Wilson loops at various N.
Other observables follow the same scaling law.

Open Polyakov line as order parameter for transl. invariance:

$$\text{Phases} \left\{ \begin{array}{cc} \beta < 0.35 \text{ (strong coupling)} & \textit{symmetric} \\ \text{intermediate} & \textit{broken} \\ \text{large } \beta & \textit{symmetric} \end{array} \right.$$

Transition line intermediate–weak : $\beta_c(N) \propto N^2$

$\longrightarrow$ **Double Scaling Limit $\beta \propto \sqrt{N}$ leads to broken phase, IR stable. Photon may survive in a NC world.**

• Here and in $\lambda\phi^4$ model : Strong IR effects due to short-range non-locality
$\Rightarrow$ UV/IR mixing persists as a non-perturbative effect in NC field theory.