Deformation Quantization of Principal Fibre Bundles

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Based on a joint work together with

M. Bordemann, N. Neumaier, S. Waldmann

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Outline:

- Motivation: (Classical) gauge theories on non-commutative space-times
- Deformation quantization of principal fibre bundles
- Deformation quantization of associated vector bundles
- Outlook

Non-commutative (NC) space-times

QM,GR:

Space-time uncertainties at the Planck-scale $\lambda_P \cong 1.6 \times 10^{-33}\mathrm{cm}$

Doplicher, Fredenhagen, Roberts 1995:

- Uncertainty relations:

  $\Delta x^0 \sum_{i=1}^{3} \Delta x^i \geq \lambda_P^2$ and $\sum_{j<k=1}^{3} \Delta x^j \Delta x^k \geq \lambda_P^2$

- Commutation relations: $[q^\mu, q^\nu] = \mathrm{i}\lambda_P^2 \Theta^{\mu\nu}$

NC geometry:

Space-time $M \leftrightarrow (C^\infty(M), \cdot) \rightsquigarrow (\mathcal{A}, \mu)$ associative NC algebra

Deformation quantization (DQ) and NC space-times

DQ: $(C^\infty(M),\cdot) \rightsquigarrow (C^\infty(M)[[\lambda]],\star)$

Star product: $a\star b = a\cdot b + \sum_{r=1}^{\infty}\lambda^r\mu_r(a,b)$

- Associative deformation of the point-wise product with bidifferential maps μ_r
- $[a,b]_\star = a\star b - b\star a = \mathrm{i}\lambda\{a,b\}_\pi + O(\lambda^2)$ with Poisson structure $\pi\in\Gamma^\infty(\Lambda^2 TM)$, locally: $\pi = \pi^{ij}\partial_i\wedge\partial_j$

NC space-time:

- Local coordinate functions: $[x^i,x^j]_\star = \mathrm{i}\lambda\pi^{ij} + O(\lambda^2)$
- $\lambda\sim\lambda_P^2$ Planck-area as deformation parameter, classical limit, analogy to QM: $\lambda_P^2\sim\hbar$

Gauge theories on NC space-times

Precondition:
NC space-time $(C^\infty(M)[[\lambda]], \star)$

Aim:
Global geometric formulation of corresponding classical gauge theories

Method:
Deformation of all algebraic structures involving the non-commutative space-time algebra $(C^\infty(M)[[\lambda]], \star)$:

- Persistence of all algebraic structures
- Classical (commutative) situation in lowest order $\lambda = 0$

Gauge theories on NC space-times

Classical gauge theory:

- $\mathsf{p} : P \longrightarrow M$ principal fibre bundle (PFB) with structure Lie group G and right action $\mathsf{r} : P \times G \longrightarrow P$
- $\pi : G \times V \longrightarrow V$ representation of G on a finite dimensional vector space V inducing an associated vector bundle $E \longrightarrow M$

Resulting problem:

Deformation quantization of principal fibre bundles

Algebraic structure of PFB

Serre-Swan:

The sections $\Gamma^\infty(E)$ are a $C^\infty(M)$-right module (finitely generated and projective)
$\rightsquigarrow$ Deformation quantization of vector bundles

Isomorphism of $C^\infty(M)$-right modules:

$$\Gamma^\infty(E) \cong (C^\infty(P) \otimes V)^G$$

Underlying algebraic structure:

The functions $C^\infty(P)$ are a G-invariant $C^\infty(M)$-right module by $f \cdot \mathsf{p}^* a$ for $f \in C^\infty(P)$ and $a \in C^\infty(M)$.

$$g \triangleright (f \cdot \mathsf{p}^* a) = \mathsf{r}_g^*(f \cdot \mathsf{p}^* a) = (\mathsf{r}_g^* f) \cdot \mathsf{p}^* a = (g \triangleright f) \cdot \mathsf{p}^* a \quad \forall g \in G$$

Deformation quantization of PFB

Definition

Let $\mathsf{p} : P \longrightarrow M$ be a PFB with structure group G and let $\star$ be a star product on M.

i) A DQ of the PFB is a G-invariant $(C^\infty(M)[[\lambda]], \star)$-right module structure $\bullet$ of $C^\infty(P)[[\lambda]]$ such that

$$f \bullet a = f \cdot \mathsf{p}^* a + \sum_{r=1}^{\infty} \lambda^r \rho_r(f, a)$$

for all $f \in C^\infty(P)$ and $a \in C^\infty(M)$ with bidifferential operators ρ_r.

Explicitly:

Right module: $f \bullet (a \star b) = (f \bullet a) \bullet b$

G-invariance: $g \triangleright (f \bullet a) = (g \triangleright f) \bullet a$

Deformation quantization of PFB

Definition

ii) Two such deformations $\bullet$ and $\tilde{\bullet}$ are called equivalent, if there exists a formal series

$$T = \mathrm{id} + \sum_{r=1}^{\infty} \lambda^r T_r$$

of G-invariant differential operators $T_r \in \mathrm{DiffOp}(C^\infty(P))^G$,

$$g \triangleright T_r = \mathsf{r}_g^* \circ T_r \circ \mathsf{r}_{g^{-1}}^* = T_r \quad \forall r \geq 1, \forall g \in G,$$

such that

$$T(f \bullet a) = T(f) \,\tilde{\bullet}\, a$$

for all $f \in C^\infty(P)[[\lambda]]$ and $a \in C^\infty(M)[[\lambda]]$.

Deformation quantization of PFB

Theorem (M. Bordemann, N. Neumaier, S. Waldmann, S. W.)

Every principal fibre bundle $\mathrm{p} : P \longrightarrow M$ *with a star product* $\star$ *on* M *admits a deformation quantization which is unique up to equivalence.*

Especially in the symplectic case: Fedosov-like construction

The commutant of the right module structure

Classical commutant within the differential operators:

$$\mathrm{DiffOp}_{\mathrm{ver}}(P) = \{D \in \mathrm{DiffOp}(C^\infty(P)) \mid D(f \cdot \mathsf{p}^* a) = D(f) \cdot \mathsf{p}^* a\}$$

G-invariant bimodule structure of $C^\infty(P)$:

$$(\mathrm{DiffOp}_{\mathrm{ver}}(P), \circ) \quad C^\infty(P) \quad (C^\infty(M), \cdot)$$

$$g \triangleright (D(f)) = (g \triangleright D)(g \triangleright f)$$

$(\mathrm{DiffOp}_{\mathrm{ver}}(P), \circ)$ is a G-invariant algebra:

$$g \triangleright (D_1 \circ D_2) = (g \triangleright D_1) \circ (g \triangleright D_2)$$

The commutant of the right module structure

Theorem (M. Bordemann, N. Neumaier, S. Waldmann, S. W.)

Let $\mathsf{p} : P \longrightarrow M$ *be a PFB and let* $\bullet$ *be a DQ with respect to* $\star$. *Then there exists a bimodule structure*

$$(\mathrm{DiffOp}_{\mathrm{ver}}(P)[[\lambda]], \star') \; {}^{(\bullet', C^\infty(P)[[\lambda]], \bullet)} \; (C^\infty(M)[[\lambda]], \star)$$

with G*-invariant deformations* $\star'$ *and* $\bullet'$, *i.e.*

$$A \star' B = A \circ B + \sum_{r=1}^\infty \lambda^r \mu'_r(A, B),$$

$$A \bullet' f = A(f) + \sum_{r=1}^\infty \lambda^r \rho'_r(A, f)$$

for all $A, B \in \mathrm{DiffOp}_{\mathrm{ver}}(P)$, $f \in C^\infty(P)$ *with bilinear* ρ'_r *and* μ'_r.

The commutant of the right module structure

Theorem (M. Bordemann, N. Neumaier, S. Waldmann, S. W.)

$$(\mathrm{DiffOp}_{\mathrm{ver}}(P)[[\lambda]], \star') \quad (\bullet', C^\infty(P)[[\lambda]], \bullet) \quad (C^\infty(M)[[\lambda]], \star)$$

Furthermore:

- *$\star'$ is unique up to equivalence*
- *Acting by $\bullet'$ and $\bullet$ the two algebras $(\mathrm{DiffOp}_{\mathrm{ver}}(P)[[\lambda]], \star')$ and $(C^\infty(M)[[\lambda]], \star)$ are mutual commutants*

Deformation quantization of vector bundles

Bursztyn, Waldmann 2000:

$E \longrightarrow M$ vector bundle:

$$(\Gamma^\infty(\mathrm{End}(E))[[\lambda]], \star'_E) \overset{(\bullet'_E, \Gamma^\infty(E)[[\lambda]], \bullet)}{} (C^\infty(M)[[\lambda]], \star)$$

- DQ: $(C^\infty(M)[[\lambda]], \star)$-right module structure $\bullet$ of $\Gamma^\infty(E)[[\lambda]]$ (Serre-Swan)
- Acting by $\bullet'_E$ and $\bullet$ the two algebras $(\Gamma^\infty(\mathrm{End}(E))[[\lambda]], \star'_E)$ and $(C^\infty(M)[[\lambda]], \star)$ are mutual commutants
- $\star'_E$ is unique up to equivalence

Deformation quantization of associated vector bundles

$$(\bullet', C^\infty(P)[[\lambda]], \bullet)$$

$$(\mathrm{DiffOp}_{\mathrm{ver}}(P)[[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

Representation $\pi : G \times V \longrightarrow V$ induces extension:

$$(\bullet', (C^\infty(P) \otimes V) \; [[\lambda]], \bullet)$$

$$((\mathrm{DiffOp}_{\mathrm{ver}}(P) \otimes \mathrm{End}(V)) \; [[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

Deformation quantization of associated vector bundles

$$(\bullet', C^\infty(P)[[\lambda]], \bullet)$$

$$(\mathrm{DiffOp}_{\mathrm{ver}}(P)[[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

G-invariance of $\star'$ and $\bullet'$:

$$(\bullet', (C^\infty(P) \otimes V)^G[[\lambda]], \bullet)$$

$$((\mathrm{DiffOp}_{\mathrm{ver}}(P) \otimes \mathrm{End}(V))^G[[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

Deformation quantization of associated vector bundles

$$(\bullet', C^\infty(P)[[\lambda]], \bullet)$$

$$(\mathrm{DiffOp}_{\mathrm{ver}}(P)[[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

$$(\bullet', (C^\infty(P) \otimes V)^G[[\lambda]], \bullet)$$

$$((\mathrm{DiffOp}_{\mathrm{ver}}(P) \otimes \mathrm{End}(V))^G[[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

$$(\bullet'_E, \Gamma^\infty(E)[[\lambda]], \bullet)$$

$$(\Gamma^\infty(\mathrm{End}(E))[[\lambda]], \star'_E) \qquad (C^\infty(M)[[\lambda]], \star)$$

Deformation quantization of associated vector bundles

$$(\bullet', C^\infty(P)[[\lambda]], \bullet)$$

$$(\mathrm{DiffOp}_{\mathrm{ver}}(P)[[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

$$(\bullet', (C^\infty(P) \otimes V)^G[[\lambda]], \bullet)$$

$$((\mathrm{DiffOp}_{\mathrm{ver}}(P) \otimes \mathrm{End}(V))^G[[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

$$\phi \downarrow$$

$$(\bullet'_E, \Gamma^\infty(E)[[\lambda]], \bullet)$$

$$(\Gamma^\infty(\mathrm{End}(E))[[\lambda]], \star'_E) \qquad (C^\infty(M)[[\lambda]], \star)$$

ϕ: surjective algebra homomorphism with

$$D \bullet' s = \phi(D) \bullet'_E s \quad \text{for all } s \in \Gamma^\infty(E)[[\lambda]]$$

Deformation quantization of associated vector bundles

$$(\bullet', C^\infty(P)[[\lambda]], \bullet)$$

$$(\mathrm{DiffOp}_{\mathrm{ver}}(P)[[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

$$(\bullet', (C^\infty(P) \otimes V)^G[[\lambda]], \bullet)$$

$$((\mathrm{DiffOp}_{\mathrm{ver}}(P) \otimes \mathrm{End}(V))^G[[\lambda]], \star') \qquad (C^\infty(M)[[\lambda]], \star)$$

$$\phi \downarrow$$

$$(\bullet'_E, \Gamma^\infty(E)[[\lambda]], \bullet)$$

$$(\Gamma^\infty(\mathrm{End}(E))[[\lambda]], \star'_E) \qquad (C^\infty(M)[[\lambda]], \star)$$

Deformation quantization of associated vector bundles

Theorem (M. Bordemann, N. Neumaier, S. Waldmann, S. W.)

i) *A deformation quantization of a principal fibre bundle* $P \longrightarrow M$ *induces a deformation quantization of every associated vector bundle* $E \longrightarrow M$*.*

ii) *In addition, there exists a surjective algebra homomorphism from the commutant of* P *onto the one of* E*.*

Outlook

Gauge Theory:

- Infinitesimal gauge transformations, gauge algebra
  $\rightsquigarrow$ SEIBERG-WITTEN maps, enveloping algebra
- Gauge fields $\rightsquigarrow$ Differential forms on PFB
- Dynamics $\rightsquigarrow$ Jet-bundles

References

- M. Bordemann, N. Neumaier, S. Waldmann, S. Weiss: *Deformation Quantization of Surjective Submersions and Principal Fibre Bundles* arXiv: 0711.2965v1 [math QA]
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