Harmonic analysis of local operators in massless free field theory

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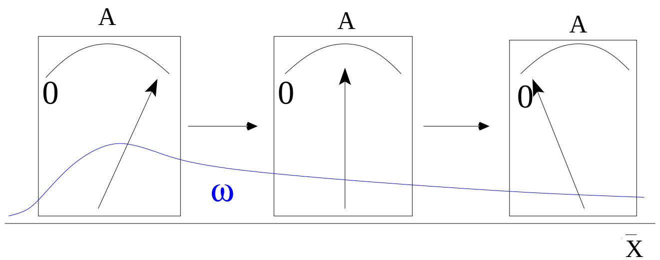


Overview

1. Particle content of QM
   $(e^{itH}, \mathcal{H})$, $\mathcal{H} = \mathcal{H}_{\rm pp} \oplus \mathcal{H}_{\rm ac} \oplus \mathcal{H}_{\rm sc}, \quad \mathcal{H}_{\rm sc} = \{0\}$.
2. Particle content of QFT
   $(\alpha_{\vec{x}},\ \mathfrak{A})$.
3. Decomposition $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O})_{\rm pp} \oplus \mathfrak{A}(\mathcal{O})_{\rm pc} \oplus \mathfrak{A}(\mathcal{O})_{\rm si}$.
4. Main result: $\mathfrak{A}(\mathcal{O})_{\rm pc}$ has finite dimension in massless free field theory.
5. Idea of the proof: harmonic analysis of pointlike-localized fields.
6. Conclusions.

1. Particle content of QM:

Example: two body scattering, short-range interaction.

$$H = H_0 + V,$$

$$\mathcal{H} = \mathcal{H}_{\mathrm{pp}} \oplus \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}},$$

$$\Omega^{\pm} = \lim_{t \to \mp\infty} e^{itH} e^{-itH_0} P_{\mathrm{ac}}.$$

The theory has a complete particle interpretation if $\mathrm{Ran}\, \Omega^{\pm} = \mathcal{H}_{\mathrm{ac}}$, and $\mathcal{H}_{\mathrm{sc}} = \{0\}$.

2. Particle content of QFT: A model independent algorithm for extracting the particle content of a theory, based on Araki-Haag detectors:

$$Q = \int d^3x \, \alpha_{\vec{x}}(C), \quad C\text{-suitable element of } \mathfrak{A}.$$

[Buchholz, Porrmann, Stein 91]

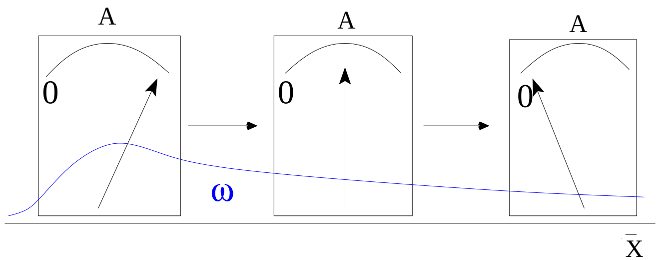
Question: Which properties of $C \in \mathfrak{A}$ control the properties of Q?

Strategy: Analysis of $(\alpha_{\vec{x}}, \mathfrak{A})$: $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O})_{\mathrm{pp}} \oplus \mathfrak{A}(\mathcal{O})_{\mathrm{pc}} \oplus \mathfrak{A}(\mathcal{O})_{\mathrm{si}}$.

3. Decomposition: $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O})_{\rm pp} \oplus \mathfrak{A}(\mathcal{O})_{\rm c}$

$$\begin{aligned} \mathfrak{A}(\mathcal{O})_{\rm pp} &:= \{ \lambda I \,|\, \lambda \in \mathbb{C} \}, \\ \mathfrak{A}(\mathcal{O})_{\rm c} &:= \{ A \in \mathfrak{A}(\mathcal{O}) \,|\, \omega_0(A) = 0 \}. \end{aligned}$$

How to decompose $\mathfrak{A}(\mathcal{O})_{\rm c}$? Decay of $\omega(\alpha_{\vec{x}}(A))$, $\omega \in S_E$.



$$\mathfrak{A}(\mathcal{O})_{\rm si} := \{ A \in \mathfrak{A}(\mathcal{O})_{\rm c} \,|\, \forall_{E \geq 0} \sup_{\omega \in S_E} \int d^3x |\omega(\alpha_{\vec{x}} A)|^2 < \infty \}.$$

Remark: In massive free field theory $\mathfrak{A}(\mathcal{O})_{\rm c} = \mathfrak{A}(\mathcal{O})_{\rm si}$.

Harmonic analysis of $\mathfrak{A}(\mathcal{O})_{\mathrm{c}}$:

$$\widetilde{A}(\vec{p}) := (2\pi)^{-\frac{3}{2}} \int e^{-i\vec{p}\vec{x}} \alpha_{\vec{x}}(A)\, d^3x.$$

- In any relativistic QFT

$$\sup_{\substack{\omega\in S_E \\ A\in\mathfrak{A}(\mathcal{O})_1}} \int d^3p\, |\vec{p}|^{4+\varepsilon} |\omega(\widetilde{A}(\vec{p}))|^2 < \infty,$$

  [Buchholz 90]. Seems too conservative at $\vec{p} = 0$.

- An operator A is persistent of order $n \geq 0$ iff $\forall_{\varepsilon>0}$

$$\sup_{\omega\in S_E} \int d^3p\, |\vec{p}|^{n+\varepsilon} |\omega(\widetilde{A}(\vec{p}))|^2 < \infty,$$

$$\sup_{\omega\in S_E} \int d^3p\, |\vec{p}|^{n-\varepsilon} |\omega(\widetilde{A}(\vec{p}))|^2 = \infty.$$

4. Main result: In massless free field theory

- $\mathfrak{A}(\mathcal{O})$ contains persistent operators of orders 0, 1, 2.
- $\mathfrak{A}(\mathcal{O})_{\mathrm{si}}$ is of finite co-dimension.
- There exists $\mathfrak{A}(\mathcal{O})_{\mathrm{pc}}$ s.t. $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O})_{\mathrm{pp}} \oplus \mathfrak{A}(\mathcal{O})_{\mathrm{pc}} \oplus \mathfrak{A}(\mathcal{O})_{\mathrm{si}}$.

5. Idea of the proof:

(a) Computation of the field content of the theory.

(b) Harmonic analysis of the fields ϕ_i.

(c) Extraction of corresponding properties of $\mathfrak{A}(\mathcal{O})$.
(Phase space structure).

5. Idea of the proof:

(a) Computation of the field content of the theory:
$\mathrm{Span}\{1, \phi, :\phi^2:, :\phi^3:, \partial_j\phi, \pi, \partial_j\partial_k\phi, \partial_j\pi, \ldots\}$

(b) Harmonic analysis of the fields ϕ_i

- 1 belongs to the point spectrum.
- ϕ is persistent of order 2.
- $:\phi^2:$ is persistent of order 1.
- $:\phi^3:$ is persistent of order 0.
- All other ϕ_i are square-integrable.

(c) Extraction of corresponding properties of $\mathfrak{A}(\mathcal{O})$.

Phase space condition [Bostelmann 98]:

There exist $\tau_i \in \mathfrak{A}(\mathcal{O})^*$ s.t. for any $A \in \mathfrak{A}(\mathcal{O})$

$$A = \sum_{i=0}^{\infty} \tau_i(A)\phi_i.$$

(Convergence in a suitable topology).

Theorem:

There exists a map $\Theta : \mathfrak{A}(\mathcal{O}) \to Q_B$ s.t. for any $A \in \mathfrak{A}(\mathcal{O})$

$$A = \omega_0(A)I + \tau_1(A)\phi + \tau_2(A) :\!\phi^2\!: + \tau_3(A) :\!\phi^3\!: + \Theta(A),$$

$\Theta(A)$ is square-integrable: $\sup_{\omega \in S_E} \int d^3x \, |\omega(\alpha_{\vec{x}}\Theta(A))|^2 < \infty$.

Corollaries of the formula

$$A = \omega_0(A)I + \tau_1(A)\phi + \tau_2(A) :\phi^2: + \tau_3(A) :\phi^3: + \Theta(A),$$

where $\Theta(A)$ is square-integrable.

- $\mathfrak{A}(\mathcal{O})$ - local algebra.
  $\mathfrak{A}(\mathcal{O})_{\rm si} \supset \ker\omega_0 \cap \ker\tau_1 \cap \ker\tau_2 \cap \ker\tau_3$.

- $\mathfrak{A}_{\rm e}(\mathcal{O})$ - even part of $\mathfrak{A}(\mathcal{O})$.
  $\mathfrak{A}_{\rm e}(\mathcal{O})_{\rm si} = \ker\omega_0 \cap \ker\tau_2$.

- $\mathfrak{A}_{\rm d}(\mathcal{O})$ - subalgebra of $\mathfrak{A}(\mathcal{O})$ generated by $\{\pi, \partial_1\phi, \partial_2\phi, \partial_3\phi\}$.
  $\mathfrak{A}_{\rm d}(\mathcal{O})_{\rm si} = \ker\omega_0$.

6. Conclusions:

- We found a decomposition $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O})_{\rm pp} \oplus \mathfrak{A}(\mathcal{O})_{\rm pc} \oplus \mathfrak{A}(\mathcal{O})_{\rm si}$. The subspaces differ in their behaviour under translations $\alpha_{\vec{x}}$.

- $\mathfrak{A}(\mathcal{O})_{\rm pc}$ originates from the algebraic structure $(\phi, :\!\phi^2\!:, \ldots)$.

- finite dimensional (in massless free field theory).

- at the boundary between point- and continuous spectrum.

- Question: Does there exist a general condition for existence, triviality or finite dimension of $\mathfrak{A}(\mathcal{O})_{\rm pc}$?