On the Path Integral in a Quantum Field Theory (QFT) on Non–Commutative (NC) Spacetime

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Outline of Talk

I) Introduction to QFT on non-commutative (NC) spacetime

II) Particular set–up of QFT on NC spacetime

III) Path integral (Hamiltonian approach):
a) in time/space NC QFT corresponding to $T^\star$-ordering
b) in time/space NC QFT corresponding to T-ordering

IV) Summary & Outlook

I) Introduction to QFT on NC spacetime

- 1)Snyder 1947, Yang 1947:
  Hope to remove divergences by introducing a minimum length
- 2)Gedankenexperiment:
  (Doplicher, Fredenhagen, Roberts: [DFR 1995])
  Creation of micro–black holes in scattering events with high energy transfer restricts possible resolution of spacetime events.
  Below Planck scale, measurements become meaningless.
  $\Longrightarrow$ spacetime uncertainty relations
- 3)String theory:
  (Connes, Douglas, Schwarz; Schomerus ; Seiberg, Witten (1998/ 1999))
  Low energy limit of open string attached to a
  D–brane in a constant background magnetic
  field can be desribed by QFT on NC *space* (, not NC *spacetime*).

II) Particular set–up of QFT on NC spacetime (1)

Popular idea to implement non–commutative structure :

- Use Weyl–Moyal correspondence & replace the pointwise defined product of functions by Moyal–product ($*$–product):

$$(f_1 * f_2)(x) := \left[\exp\left(\tfrac{i}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu\right)f_1(x)f_2(y)\right]_{y=x}$$

- $[\hat{x}_\mu, \hat{x}_\nu] =: i\theta_{\mu\nu}1$; $\hat{x}_\mu$, $\hat{x}_\nu$: coordinate operators; $\theta_{\mu\nu}$: real, antisymmetric, constant matrix $(d = 1+3)$
- trace property: $\int d^4x\, f * g(x) = \int d^4x\, f \cdot g(x)$ for $f, g \,\epsilon\, \mathcal{S}(\mathbb{R}^{3+1})$
- here: time/space non–commutativity $(\theta^{0i} \neq 0,\ i = 1, 2, 3)$

II) Particular set–up of QFT on NC spacetime (2)

Starting point for QFT on NC spacetime

- Consider, e. g., free part of action (neutral massive scalar fields):

$$S_{kin}^{NC} = \frac{1}{2} \int d^4x : (\frac{\partial}{\partial t}\phi * \frac{\partial}{\partial t}\phi)(x) : + : (\partial_i\phi * \partial^i\phi)(x) : + m^2 : (\phi * \phi)(x) : = S_{kin}$$

- due to trace property of star product

  $\Longrightarrow$ free QFT in NC case equals free (ordinary) QFT

- Consider then interaction part of action:

$$S_I^{NC} = \frac{1}{2} \int d^4x \lambda : (\phi * \ldots * \phi)(x) : = \frac{1}{2}\lambda \int d^4k_1 \ldots \int d^4k_n : \check{\phi}(k_1)\ldots\check{\phi}(k_n) : e^{\frac{-i}{2}\sum\limits_{i<j} k_i^\mu \theta_{\mu\nu} k_j^\nu} \delta^4(\sum k_i)$$

- $\Longrightarrow$ Perturbation theory (generally): Vertices contain trigonometric functions of momenta

IIIa) Path Integral corresponding to $T^\star$–Ordering (1)

- Naïve ansatz for path integral ($\theta_{0i} \neq 0$, $i \,\epsilon\, \{1, 2, 3\}$):

  Start from a *nonlocal* interaction Hamiltonian density

  $$\mathcal{H}_{int}(\phi)_* := \phi * \phi * \phi(x), \text{ e. g.,}$$

  and plug it in the formula for generating functional of *local* case ($\Delta_c(z)$: causal propagator):

  $$Z[J] = \exp[-i \int d^4 z \mathcal{H}_{int}(\tfrac{\delta}{i\delta J(z)})_*] \times \\ \times \exp[\tfrac{-1}{2} \int d^4 a \int d^4 b J(a) \Delta_c(a-b) J(b)]$$

- Perturbative expansion leads to naïve Feynman rules: Graphs with causal propagators as internel lines and vertices that are multiplied by *trigonometric functions* of momenta

- Example (fishgraph in momentum space):

$$\left(\frac{i}{p^2 - m^2 + i\epsilon}\right)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \cos^2\left(\frac{p_\mu \theta^{\mu\nu} q_\nu}{2}\right) \frac{i}{(p-q)^2 - m^2 + i\epsilon}$$

IIIa) Path Integral corresponding to $T^\star$–Ordering (2)

- These Feynman rules are also obtained by starting from Gell–Mann - Low formula (canonical approach = operator approach) and applying $T^\star$-operator.
- Gell-Mann Low formula with $\mathrm{T}^\star - \textit{product}$:

$$\langle\Omega|\,T^\star\{\Phi_\theta(x)\Phi_\theta(y)\}\,|\Omega\rangle = \frac{\langle 0|\,T^\star\{\phi_I(x)\phi_I(y)exp[i\int\, d^4x\mathcal{H}_{int}(\phi_I(x))_*]\}\,|0\rangle}{\langle 0|\,T^\star\{exp[i\int\, d^4x\mathcal{H}_{int}(\phi_I(x))_*]\}\,|0\rangle}$$

Φ_θ: Heisenberg field; $|\Omega\rangle$: ground state of interacting theory; $|0\rangle$: ground state of free theory; ϕ_I: Dirac picture field

- $T^\star$–product: all time derivatives of star product act after time ordering, see Heslop & Sibold [11/04].
- According to Gomis & Mehen [02,00]: Feynman rules violate unitarity.
- See K. Fujikawa [06/04]: path integral from equation of motion (not Hamiltonian approach !): same Feynman rules as above

IIIa) Path Integral corresponding to $T^\star$–Ordering (3)

- Example (fishgraph in position space):

$$T^{(2)}_\theta(z) := \int \frac{d^4p\, e^{-iz\cdot p}}{(2\pi)^4}\Big(\frac{i}{p^2-m^2+i\epsilon}\Big)^2 \int \frac{d^4q}{(2\pi)^4}\frac{(i)^2\cos^2\big(\frac{p_\mu\theta^{\mu\nu}q_\nu}{2}\big)}{(q^2-m^2+i\epsilon)((p-q)^2-m^2+i\epsilon)},$$

then with $\Lambda \,\epsilon\, SO(1,3)$: $\boxed{T^{(2)}_\theta(\Lambda z) = T^{(2)}_{\Lambda^{-1}\theta(\Lambda^{-1})^\tau}(z)}$

- continuous part of the spectral representation for the fishgraph:

$$\sigma^{(2)}_\theta(p^2,\tilde p^2) = \frac{\vartheta(p^2-4m^2)\vartheta(p^0)\gamma(p^2)}{64\pi^2(p^2-m^2)^2}\left(1+\frac{\sin\big(\frac{\gamma(p^2)}{2}\sqrt{-p^2(\tilde p)^2}\big)}{\frac{\gamma(p^2)}{2}\sqrt{-p^2(\tilde p)^2}}\right)$$

with $\tilde p^\mu := p_\nu\theta^{\mu\nu}$, $\gamma(p^2) := \sqrt{1-\frac{4m^2}{(p)^2}}$ and $p^2 := p_0^2 - |\vec p|^2$.

IIIa) Path Integral corresponding to $T^{\star}$–Ordering (4)

- However: Causal (or rather chronological) time ordering is lost (C. Dehne, to appear)!
- Example for causal time ordering:

$$\langle\Omega|T\Phi_\theta(x_1)\Phi_\theta(x_2)\Phi_\theta(x_3)|\Omega\rangle =$$
$$\langle\Omega|\Phi_\theta(x_1)\Phi_\theta(x_2)\Phi_\theta(x_3)|\Omega\rangle, \text{ if } x_1^0 > x_2^0 > x_3^0,$$
$$\langle\Omega|\Phi_\theta(x_1)\Phi_\theta(x_3)\Phi_\theta(x_2)|\Omega\rangle, \text{ if } x_1^0 > x_3^0 > x_2^0,$$
$$\langle\Omega|\Phi_\theta(x_2)\Phi_\theta(x_1)\Phi_\theta(x_3)|\Omega\rangle, \text{ if } x_2^0 > x_1^0 > x_3^0,$$
$$\langle\Omega|\Phi_\theta(x_2)\Phi_\theta(x_3)\Phi_\theta(x_1)|\Omega\rangle, \text{ if } x_2^0 > x_3^0 > x_1^0,$$
$$\langle\Omega|\Phi_\theta(x_3)\Phi_\theta(x_1)\Phi_\theta(x_2)|\Omega\rangle, \text{ if } x_3^0 > x_1^0 > x_2^0,$$
$$\langle\Omega|\Phi_\theta(x_3)\Phi_\theta(x_2)\Phi_\theta(x_1)|\Omega\rangle, \text{ if } x_3^0 > x_2^0 > x_1^0$$

- In the time/space NC case, vertex becomes fuzzy and decompostion into six different terms is not possible, e. g.:

$$\frac{g}{3!}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{(i)^3 e^{-i(p_1\cdot x_1+p_2\cdot x_2+p_3\cdot x_3)}}{p_1^2-m^2+i\epsilon}\frac{\cos(p_1^\mu\theta_{\mu\nu}p_2^\nu)}{p_2^2-m^2+i\epsilon}\frac{(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)}{p_3^2-m^2+i\epsilon}$$

IIIb) Path Integral corresponding to T–Ordering (1)

- Question: In the case of $\theta_{0i} \neq 0$ ($i \,\epsilon \,\{1,2,3\}$), can one derive a generating functional (or modify the usual one in such a way) that the resulting Feynman rules preserve *causal time ordering*?

- Answer: Yes!

$$\boxed{\begin{aligned} Z[J] = \exp[-i \int d^4 z [\mathcal{H}_{int}(\tfrac{\delta}{i\delta J(z)})_*]_{\overrightarrow{\theta}}]\times \\ \times \exp[\tfrac{-1}{2} \int d^4 a \int d^4 b J(a) T\Delta_+(a-b) J(b)] \end{aligned}}$$

- $T\Delta_\pm(z) := \vartheta(z^0)\Delta_+(z) + \vartheta(-z^0)\Delta_+(-z) = \Delta_c(z)$,

  $\Delta_+(z) := \int \frac{d^4 p}{(2\pi)^3} e^{-ip\cdot z} \vartheta(p^0)\delta(p^2 - m^2)$

- $[(\frac{\delta}{\delta J(x)})_*]_{\overrightarrow{\theta}}$: For each time–ordered configuration take first the time derivative (associated to θ_{0i}) of $\Delta_+(x)$. Then, realize the time ordering by multiplication with step function. (The argument of the step function never contains θ_{0i}).

IIIb) Path Integral corresponding to T–Ordering (2)

Main result: Feynman rules are the same as those derived within the canonical (operator) approach and leading to old–fashioned perturbation theory (TOPT) (equivalence between canonical approach and path integral)

- Old–fashioned perturbation theory (TOPT) (Liao & Sibold [05/02], [06/02]; see also Liao & Dehne [11/02]): Start from Gell–Mann - Low formula and apply T–ordering.
- Gell-Mann Low formula with T–product:

$$\langle\Omega|T\{\Phi_\theta(x)\Phi_\theta(y)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_I(x)\phi_I(y)exp[i\int d^4x\mathcal{H}_{int}(\phi_I(x))_*]\}|0\rangle}{\langle 0|T\{exp[i\int d^4x\mathcal{H}_{int}(\phi_I(x))_*]\}|0\rangle}$$

Φ_θ: Heisenberg field; $|\Omega\rangle$: ground state of interacting theory; $|0\rangle$: ground state of free theory; ϕ_I: Dirac picture field

- T–product: all time derivatives of star product act before time ordering is applied (See also Fujikawa [06/04], [10/04]; Heslop & Sibold [11/04].)

IIIb) Path Integral corresponding to T–Ordering (3)

- Perturbative example:

$$\int d^4z \langle 0|T\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I * \phi_I * \phi_I(z)|0\rangle$$

$$= \int \frac{d^4p_1 e^{-ip_1\cdot x_1}}{(2\pi)^4 2\omega_{\vec{p}_1}} \frac{d^4p_2 e^{-ip_2\cdot x_2}}{(2\pi)^4 2\omega_{\vec{p}_2}} \frac{d^4p_3 e^{-ip_3\cdot x_3}}{(2\pi)^4 2\omega_{\vec{p}_3}} \sum_{\lambda_1,\lambda_2,\lambda_3\,\epsilon\,\{-,+\}} \times$$

$$\times \sum_{\sigma\,\epsilon\,P_3} \frac{(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\exp[i(p_{\sigma(1)\lambda\sigma(1)},p_{\sigma(2)\lambda\sigma(2)},p_{\sigma(3)\lambda\sigma(3)})]}{6(\lambda_1 p_1^0-\omega_{\vec{p}_1}+i\epsilon)(\lambda_2 p_2^0-\omega_{\vec{p}_2}+i\epsilon)(\lambda_3 p_3^0-\omega_{\vec{p}_3}+i\epsilon)}$$

$$= \frac{(-1)^3}{6}\int d^4z \sum_{\sigma\,\epsilon\,P_3}\left[\exp[i(\tfrac{\partial}{\partial x_{\sigma(1)}},\tfrac{\partial}{\partial x_{\sigma(2)}},\tfrac{\partial}{\partial x_{\sigma(3)}})]\right]_\theta^{\rightarrow} \times$$

$$\times\Delta_c(x_1-z)\Delta_c(x_2-z)\Delta_c(x_3-z),$$

where $(a,b,c) := (a\wedge b + a\wedge c + b\wedge c)$, $a\wedge b := \frac{1}{2}a^\mu\theta_{\mu\nu}b^\nu$, $p_{i\lambda_i} := (\lambda_i\omega_{\vec{p}_i},\vec{p}_i)^\tau$ and $\omega_{\vec{p}_i} := \sqrt{|\vec{p}_i|^2+m^2}$.

- These Feynman rules maintain unitarity and by construction causal time–ordering.
- Generating functional is less tedious than operator approach!

IIIb) Path Integral corresponding to T–Ordering (4)

- Example (fishgraph in position space):

$$T_\theta^{(2)}(z) := \sum_{\lambda_{1,2}\epsilon\{-,+\}} \int \frac{d^4 p e^{-ip\cdot z}}{(2\pi)^4} \frac{d^3 q_1}{\omega_{\vec{q}_1}} \frac{d^3 q_2}{\omega_{\vec{q}_2}} \frac{\delta^{(3)}(\vec{q}_1+\vec{q}_2-\vec{p})}{4i} \frac{(\omega_{\vec{p}}+\lambda_1 p^0)}{\omega_{\vec{p}}} \frac{(\omega_{\vec{p}}+\lambda_2 p^0)}{\omega_{\vec{p}}} \times$$
$$\times \left(\frac{(\sum_{sym} e^{-i(-p_{\lambda_1},q_{1+},q_{2+})} e^{-i(-p_{\lambda_2},q_{1+},q_{2+})})}{p^0 - \omega_{\vec{q}_1} - \omega_{\vec{q}_2} + i\epsilon} + \frac{(\sum_{sym} e^{-i(-p_{\lambda_1},q_{1-},q_{2-})} e^{-i(-p_{\lambda_2},q_{1-},q_{2-})})}{-p^0 - \omega_{\vec{q}_1} - \omega_{\vec{q}_2} + i\epsilon} \right),$$

then with $\Lambda \,\epsilon\, SO(1,3)$: $\boxed{T_\theta^{(2)}(\Lambda z) \neq T^{(2)}_{\Lambda^{-1}\theta(\Lambda^{-1})^\tau}(z)}$!

- continuous part of the spectral representation for the fishgraph:

$$\boxed{\sigma_\theta^{(2)} \neq f(p^2, \tilde{p}^2)}$$

(in contrast to the case of covariant time–ordering)

- $\longrightarrow$ different quantization prescription

IV) Summary and Outlook (1)

- Main result: successful derivation of path integral formula corresponding to the T–product in canonical case (Hamiltonian approach)
- Feynman rules are identical to those of TOPT and thus preserve unitarity and causal time–ordering.
- time–ordering (or rather quantization prescription) not rigidly implemented in the path integral
- in progress: path integral based on T–operator
  I) in u–coordinates, II) starting from field equation

IV) Summary and Outlook (2)

- in progress: Wick rotation and Euclidean Feynman rules for causally time–ordered (and naïvely canonically quantized) path integral
- in this case: reflection positivity requires $\theta_{0k} \rightarrow \pm i\theta_{0k}$ $(k = 1, 2, 3)$
- in this case: one–loop diagramm for theory with $\phi * \phi * \phi * \phi(x)$–self–interaction remains finite for any configuration of external momentum p

  $\longrightarrow$ no UV/IR connection for these (new) reflection positive Euclidean Feynman rules !!!

- in progress: Wick rotation and Euclidean Feynman rules for covariantly (quantized and) time–ordered path integral

IV) Summary and Outlook (3)

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