On the UV/IR mixing problem on the noncommutative Minkowski space

Dorothea Bahns

30 November 2007

Main point

Euclidean noncommutative field theory and and Minkowskian noncommutative field theory have very different properties, especially regarding renormalization.

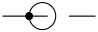
There are indications that hard ultraviolet/infrared mixing might be absent on the noncommutative Minkowski space. Certainly it would be different: absence proved for a certain class of graphs.

Main point

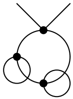
Euclidean noncommutative field theory and and Minkowskian noncommutative field theory have very different properties, especially regarding renormalization.

There are indications that hard ultraviolet/infrared mixing might be absent on the noncommutative Minkowski space. Certainly it would be different: absence proved for a certain class of graphs.

Example [Spoiler]: contrary to Euclidean ncQFT, the insertion of



into higher order graphs, e.g.



is well defined in Minkowskian ncQFT.

Quantum Field Theory on $\mathbb{R}^4$

- Starting point: partial differential equation (free field eqn)

$$(\partial_t^2 \pm \Delta + m^2)\,\varphi = 0$$

mass $m > 0$. The signature matters:
$-$ hyperbolic (Minkowskian) versus
$+$ elliptic (Euclidean) case.

Quantum Field Theory on $\mathbb{R}^4$

- Starting point: partial differential equation (free field eqn)

$$(\partial_t^2 \pm \Delta + m^2)\, \varphi = 0$$

mass $m > 0$. The signature matters:
$-$ hyperbolic (Minkowskian) versus
$+$ elliptic (Euclidean) case.

- φ is an operator valued distribution.

Quantum Field Theory on $\mathbb{R}^4$

- Starting point: partial differential equation (free field eqn)

$$(\partial_t^2 \pm \Delta + m^2)\,\varphi = 0$$

mass $m > 0$. The signature matters:
$-$ hyperbolic (Minkowskian) versus
$+$ elliptic (Euclidean) case.

- φ is an operator valued distribution.
- Interaction:

$$(\partial_t^2 \pm \Delta + m^2)\varphi = -g\,P'(\varphi)$$

P' derivative of a polynomial, g coupling constant

Operator valued distributions

The free field φ is in fact an operator valued distribution. We write

$$\varphi(f) = \int d^4x \, \varphi(x) f(x)$$

for a testfunction $f \in \mathcal{S}(\mathbb{R}^4)$.

u is an operator valued distribution provided

- it maps a Schwartzfunction $f \in \mathcal{S}(\mathbb{R}^4)$ to an operator on Fockspace $u(f)$
- such that for any two elements ψ_1, ψ_2 of Fockspace, the map

$$\mathcal{S}(\mathbb{R}^4) \ni \quad f \mapsto \langle \psi_1, u(f)\psi_2 \rangle$$

defines a tempered distribution.

Free fields

- Free field

$$\varphi(f) = \int d^4x \; \varphi(x) \, f(x)$$

- For signature $(\partial_t^2 - \Delta + m^2)$ we have:

$$\varphi(x) = \int d^3\mathbf{k} \; \omega_{\mathbf{k}}^{-1} \; (a(k) \, e^{ikx} + a^*(k) \, e^{-ikx})|_{k=(\omega_{\mathbf{k}}, \mathbf{k})}$$

- with $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$
  annihilation, creation operators $a(k), a^*(k)$ on Fock space
  and $kx = k_\mu \eta^{\mu\nu} x_\nu$ with $\eta = (+1, -1, -1, -1)$ (signature)
- Eventually, testfunctions f are removed in the formalism (adiabatic limit).

Perturbation theory

Idea: interacting theory determined by free field

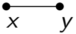
- Functional integral approach [Euclidean]
- S-matrix approach (Dyson series) [Minkowskian]
- Yang-Feldman approach (based on the field equation) [both signatures]

Euclidean Feynman graphs

- On $\mathbb{R}^4$, all perturbative setups lead to the Feynman rules.

Euclidean Feynman graphs

- On $\mathbb{R}^4$, all perturbative setups lead to the Feynman rules.
- Graphs in $\mathbb{R}^2$: vertices, edges, labelled open edges
  $\#$ edges $+$ $\#$ open edges at a given vertex determined by P
- Graphs correspond to distributions

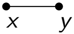


$\leftrightarrow$ $\Delta_E(x-y)$ Euclidean propagator

$\leftrightarrow$ e^{ipx} $p \in \mathbb{R}^4$ label

Euclidean Feynman graphs

- On $\mathbb{R}^4$, all perturbative setups lead to the Feynman rules.
- Graphs in $\mathbb{R}^2$: vertices, edges, labelled open edges
  \# edges + \# open edges at a given vertex determined by P
- Graphs correspond to distributions



$\leftrightarrow$ $\Delta_E(x-y)$ Euclidean propagator

$\leftrightarrow$ e^{ipx} $p \in \mathbb{R}^4$ label

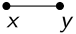
- Signature:
  Δ_E = distributional fundamental solution for $(\partial_t^2 + \Delta + m^2)$
  $px = p_\mu \eta^{\mu\nu} x_\nu$ with $\eta = (+,+,+,+)$.

Minkowskian Feynman graphs

- Graphs in $\mathbb{R}^2$: vertices, edges, labelled open edges
  # edges + # open edges at a given vertex determined by P

Minkowskian Feynman graphs

- Graphs in $\mathbb{R}^2$: vertices, edges, labelled open edges
  \# edges + \# open edges at a given vertex determined by P
- Graphs correspond to distributions

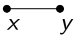


$\leftrightarrow$ $\Delta_F(x-y)$ Feynman propagator

$\leftrightarrow$ e^{ipx} $p \in \mathbb{R}^4$ label

Minkowskian Feynman graphs

- Graphs in $\mathbb{R}^2$: vertices, edges, labelled open edges
  # edges + # open edges at a given vertex determined by P
- Graphs correspond to distributions



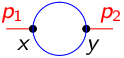
$\leftrightarrow$ $\Delta_F(x-y)$ Feynman propagator

$\leftrightarrow$ e^{ipx} $p \in \mathbb{R}^4$ label

- Signature:
  Δ_F = distributional fundamental solution for $(\partial_t^2 - \Delta + m^2)$
  $px = p_\mu \eta^{\mu\nu} x_\nu$ with $\eta = (+,-,-,-)$.

Feynman graph example

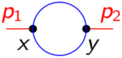
Example: fish graph [Minkowskian]



$$\longleftrightarrow \quad \int d^4x d^4y \; \Delta_F(x-y)^2 \; e^{ip_1 x} \; e^{ip_2 y} \; g(x,y)$$

Feynman graph example

Example: fish graph [Minkowskian]



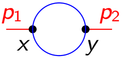
$$\leftrightarrow \quad \int d^4x d^4y \, \Delta_F(x-y)^2 \, e^{ip_1 x} \, e^{ip_2 y} \, g(x,y)$$

Same as well known momentum space calculations: remove g, replace

$$\Delta_F(x-y) = \int d^4k \underbrace{\frac{1}{k^2 - m^2 + i\epsilon}}_{=\tilde{\Delta}_F(k)} \, e^{-ik(x-y)}$$

Feynman graph example

Example: fish graph [Minkowskian]



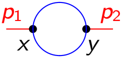
$$\longleftrightarrow \quad \int d^4x d^4y \; \Delta_F(x-y)^2 \; e^{ip_1 x} \; e^{ip_2 y} \; g(x,y)$$

Same as well known momentum space calculations: remove g, replace

$$\Delta_F(x-y) = \int d^4k \; \tilde{\Delta}_F(k) \; e^{-ik(x-y)}$$

Feynman graph example

Example: fish graph [Minkowskian]



$$\leftrightarrow \quad \int d^4x d^4y \; \Delta_F(x-y)^2 \; e^{ip_1 x} \; e^{ip_2 y} \; g(x,y)$$

Same as well known momentum space calculations: remove g, replace

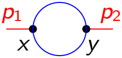
$$\Delta_F(x-y) = \int d^4k \; \tilde{\Delta}_F(k) \; e^{-ik(x-y)}$$

and perform the integrals over x and y

$$\int d^4x d^4y \int d^4k_1 d^4k_2 \; \tilde{\Delta}_F(k_1) \; \tilde{\Delta}_F(k_2) \; e^{-i(k_1+k_2)(x-y)} e^{ip_1 x} \; e^{ip_2 y}$$

Feynman graph example

Example: fish graph [Minkowskian]



$$\leftrightarrow \quad \int d^4x d^4y \; \Delta_F(x-y)^2 \; e^{ip_1 x} \; e^{ip_2 y} \; g(x,y)$$

Same as well known momentum space calculations: remove g, replace

$$\Delta_F(x-y) = \int d^4k \; \tilde{\Delta}_F(k) \; e^{-ik(x-y)}$$

and perform the integrals over x and y

$$\int d^4x d^4y \int d^4k_1 d^4k_2 \; \tilde{\Delta}_F(k_1) \; \tilde{\Delta}_F(k_2) \; e^{-i(k_1+k_2)(x-y)} e^{ip_1 x} \; e^{ip_2 y}$$

$$= \int d^4k_1 d^4k_2 \; \frac{1}{k_1^2 - m^2 + i\epsilon} \; \frac{1}{k_2^2 - m^2 + i\epsilon} \; \delta(k_1 + k_2 + p_1) \; \delta(p_1 + p_2)$$

The Feynman propagator

- What is the Feynman propagator Δ_F?

The Feynman propagator

- What is the Feynman propagator Δ_F?
- fundamental distributional solution for $(\partial_t^2 - \Delta + m^2)$.
  Signature matters!

The Feynman propagator

- What is the Feynman propagator Δ_F?
- fundamental distributional solution for $(\partial_t^2 - \Delta + m^2)$. Signature matters!
- Kernel of the distribution given by

$$\Delta_F(x) = \theta(x_0)\Delta_+(x) + \theta(-x_0)\Delta_+(-x)$$

for $x = (x_0, \mathbf{x})$

The Feynman propagator

- What is the Feynman propagator Δ_F?
- fundamental distributional solution for $(\partial_t^2 - \Delta + m^2)$.
  Signature matters!
- Kernel of the distribution given by

$$\Delta_F(x) = \theta(x_0)\Delta_+(x) + \theta(-x_0)\Delta_+(-x)$$

for $x = (x_0, \mathbf{x})$
with Heaviside step function (distribution) θ
timeorder: $\theta(x_0 - y_0) \ \leftrightarrow \ x_0$ later than y_0

The Feynman propagator

- What is the Feynman propagator Δ_F?
- fundamental distributional solution for $(\partial_t^2 - \Delta + m^2)$. Signature matters!
- Kernel of the distribution given by

$$\Delta_F(x) = \theta(x_0)\Delta_+(x) + \theta(-x_0)\Delta_+(-x)$$

for $x = (x_0, \mathbf{x})$
with Heaviside step function (distribution) θ
timeorder: $\theta(x_0 - y_0) \;\leftrightarrow\; x_0$ later than y_0
and 2-point-function (distribution) $f \mapsto \Delta_+(f)$ with kernel

$$\Delta_+(x-y) = \langle \Omega, \varphi(x)\varphi(y)\Omega \rangle$$

with φ the free field.

Connection with S-matrix formalism

In S-matrix formalism, two ingredients:

Heaviside functions θ from time order T,

2-point-functions Δ_+ from contractions of fields.

Connection with S-matrix formalism

In S-matrix formalism, two ingredients:

Heaviside functions θ from time order T,

2-point-functions Δ_+ from contractions of fields.

S-matrix:

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \, T \int dt_1 \dots dt_n \; H_I(t_1) \cdots H_I(t_n)$$

with interaction Hamiltonian $H_I(t) = \int_{x^0=t} d^3\mathbf{x} \; g \, P(\varphi)$,

time ordering T, the free field φ i.e. $(\partial_t^2 - \Delta + m^2)\,\varphi = 0$.

Connection with S-matrix formalism – fish graph

Connection with S-matrix formalism – fish graph

Example: $H_I(t) = \int_{x^0=t} d^3\mathbf{x}\; g\,\varphi^3(x)$

$$S_2 \propto \int d^4x d^4y\, \left(\theta(x_0 - y_0)\; :\varphi(x)^3::\varphi(y)^3: + (x \leftrightarrow y)\right) g(x)g(y)$$

Connection with S-matrix formalism – fish graph

Example: $H_I(t) = \int_{x^0=t} d^3\mathbf{x}\; g\,\varphi^3(x)$

$$S_2 \propto \int d^4x d^4y \left(\theta(x_0 - y_0)\; :\varphi(x)^3::\varphi(y)^3: + (x \leftrightarrow y)\right) g(x)g(y)$$

$$\langle p| :\varphi(x)^3::\varphi(y)^3: |q\rangle \;\rightarrow\; e^{-ipx}\, e^{iqy}\; \Delta_+^2(x-y) + \ldots$$

θ and Δ_+ appear such that together they yield product of Feynman propagators Δ_F.

long known – success story of Feynman over Dyson rules.

Euclidean vs. Minkowskian on $\mathbb{R}^4$

The Euclidean propagator Δ_E is distributional fundamental solution for $(\partial_t^2 + \Delta + m^2)$,

$$\Delta_E(x) = \int d^4p \, \frac{1}{p^2 + m^2} \, e^{-ikx}$$

Euclidean vs. Minkowskian on $\mathbb{R}^4$

The Euclidean propagator Δ_E is distributional fundamental solution for $(\partial_t^2 + \Delta + m^2)$,

$$\Delta_E(x) = \int d^4p \, \frac{1}{p^2 + m^2} \, e^{-ikx}$$

It is the only propagator in the theory: Δ_E is the 2-point-function – there is no time ordering via Heaviside!

Euclidean vs. Minkowskian on $\mathbb{R}^4$

The Euclidean propagator Δ_E is distributional fundamental solution for $(\partial_t^2 + \Delta + m^2)$,

$$\Delta_E(x) = \int d^4p \, \frac{1}{p^2 + m^2} \, e^{-ikx}$$

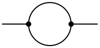
It is the only propagator in the theory: Δ_E is the 2-point-function – there is no time ordering via Heaviside!

Theorem (Osterwalder+Schrader): Perturbation theory based on Δ_E is in 1:1 relation with that based on Δ_F. They can be transformed into one another via the Wick rotation.

Excursion: Renormalization in position space

- Products of Feynman propagators Δ_F^n (in position space) are ill-defined.

Excursion: Renormalization in position space

- Products of Feynman propagators Δ_F^n (in position space) are ill-defined.
- Idea: products of Feynman propagators are well-defined distributions on testfunctions $g : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}$, not on testfunctions g with $0 \in \mathrm{supp}(g)$ (method: wavefront sets).
- Renormalization $\simeq$ extension of distributions.
- Example: , for $g \in \mathcal{S}(\mathbb{R}^4)$,

$$\int dxdy \, (\Delta_F(x-y)^2)_R \, e^{ip_1 x} \, e^{ip_2 y} \, g(x) \, g(y)$$

$$= \int dudy \, \Delta_F(u)^2 \big(e^{ip_1(u+y)} \, e^{ip_2 y} \, g(u+y) \, g(y) - w(u) \, e^{i(p_1+p_2)y} g(y) \, g(y) \big)$$

w renormalization functions (counterterms) with $w(0) = 1$, fixed by renormalization conditions.

Main Tool of renormalization

To renormalize a complicated graph

Main Tool of renormalization

To renormalize a complicated graph

take it apart into smaller graphs, renormalize those lower order graphs

Main Tool of renormalization

To renormalize a complicated graph

- take it apart into smaller graphs, renormalize those lower order graphs
- insert renormalized lower order graphs into the big graph, take care only of the remaining "overall divergences"

Main Tool of renormalization

To renormalize a complicated graph

- take it apart into smaller graphs, renormalize those lower order graphs
- insert renormalized lower order graphs into the big graph, take care only of the remaining “overall divergences”

The combinatorics is taken care of by Zimmermann’s forest formula (more recently reformulated in the framework of the Hopf algebras by Connes+Kreimer)

Noncommutative flat spacetime: M_θ and E_θ

Weyl algebra generated by e^{ikq}, $k \in \mathbb{R}^4$, $q_0, \ldots, q_3$ generators of the Heisenberg algebra, i.e.

$$e^{ikq} e^{ipq} = e^{i(k+p)q} e^{-\frac{i}{2} k\theta p} \ , \quad k, p \in \mathbb{R}^4$$

with θ antisymmetric maximal rank 4×4-matrix over $\mathbb{R}$.

Noncommutative flat spacetime: M_θ and E_θ

Weyl algebra generated by e^{ikq}, $k \in \mathbb{R}^4$, $q_0, \ldots, q_3$ generators of the Heisenberg algebra, i.e.

$$e^{ikq} e^{ipq} = e^{i(k+p)q} e^{-\frac{i}{2} k\theta p} \quad , \qquad k, p \in \mathbb{R}^4$$

with θ antisymmetric maximal rank 4×4-matrix over $\mathbb{R}$.

Signature: linear combinations $kq = k_\mu \eta^{\mu\nu} q_\nu$ with

$\eta = (1, +1, +1, +1) \quad \rightarrow \quad E_\theta$ (Euclidean)

$\eta = (1, -1, -1, -1) \quad \rightarrow \quad M_\theta$ (Minkowski)

Noncommutative flat spacetime: M_θ and E_θ

Weyl algebra generated by e^{ikq}, $k \in \mathbb{R}^4$, $q_0, \dots, q_3$ generators of the Heisenberg algebra, i.e.

$$e^{ikq} e^{ipq} = e^{i(k+p)q} e^{-\frac{i}{2} k\theta p} \quad , \quad k, p \in \mathbb{R}^4$$

with θ antisymmetric maximal rank 4×4-matrix over $\mathbb{R}$.

Signature: linear combinations $kq = k_\mu \eta^{\mu\nu} q_\nu$ with

$$\eta = (1, +1, +1, +1) \quad \to \quad E_\theta \quad \text{(Euclidean)}$$
$$\eta = (1, -1, -1, -1) \quad \to \quad M_\theta \quad \text{(Minkowski)}$$

and accordingly for the twisting.

Quantum Fields on M_θ and E_θ

1. Let S be the set of states ω on the Weyl algebra whose associated Wigner functions ψ_ω are Schwartzfunctions. The free field ϕ on the Weyl algebra E_θ or M_θ is the operator valued distribution $S \ni \omega \mapsto \phi(\omega) := \varphi(\psi_\omega)$ with φ the ordinary free field of Euclidean or Lorentz signature.

Quantum Fields on M_θ and E_θ

1. Let S be the set of states ω on the Weyl algebra whose associated Wigner functions ψ_ω are Schwartzfunctions. The free field ϕ on the Weyl algebra E_θ or M_θ is the operator valued distribution $S \ni \omega \mapsto \phi(\omega) := \varphi(\psi_\omega)$ with φ the ordinary free field of Euclidean or Lorentz signature.
2. Effective picture: define $\mathcal{S}(M_\theta)$ and $\mathcal{S}(E_\theta)$ via $(\mathcal{S}(\mathbb{R}^4), \star)$ with twisted convolution product $\star$,

$$f \star g(x) = \int d^4k d^4p \; \tilde{f}(k) \, \tilde{g}(p) \, e^{-\frac{i}{2} k\theta p} \; e^{i(k+p)x}$$

Quantum Fields on M_θ and E_θ

1. Let S be the set of states ω on the Weyl algebra whose associated Wigner functions ψ_ω are Schwartzfunctions. The free field ϕ on the Weyl algebra E_θ or M_θ is the operator valued distribution $S \ni \omega \mapsto \phi(\omega) := \varphi(\psi_\omega)$ with φ the ordinary free field of Euclidean or Lorentz signature.
2. Effective picture: define $\mathcal{S}(M_\theta)$ and $\mathcal{S}(E_\theta)$ via $(\mathcal{S}(\mathbb{R}^4), \star)$ with twisted convolution product $\star$,

$$f \star g(x) = \int d^4k d^4p \; \tilde{f}(k)\, \tilde{g}(p)\, e^{-\frac{i}{2} k\theta p} \; e^{i(k+p)x}$$

Signature enters via η in $kx = k_\mu \eta^{\mu\nu} x_\nu$ and the twisting.

Quantum Fields on M_θ and E_θ

1. Let S be the set of states ω on the Weyl algebra whose associated Wigner functions ψ_ω are Schwartzfunctions. The free field ϕ on the Weyl algebra E_θ or M_θ is the operator valued distribution $S \ni \omega \mapsto \phi(\omega) := \varphi(\psi_\omega)$ with φ the ordinary free field of Euclidean or Lorentz signature.
2. Effective picture: define $\mathcal{S}(M_\theta)$ and $\mathcal{S}(E_\theta)$ via $(\mathcal{S}(\mathbb{R}^4), \star)$ with twisted convolution product $\star$,

$$f \star g(x) = \int d^4k d^4p \; \tilde{f}(k)\, \tilde{g}(p)\, e^{-\frac{i}{2}k\theta p} \; e^{i(k+p)x}$$

   Signature enters via η in $kx = k_\mu \eta^{\mu\nu} x_\nu$ and the twisting.
3. Formally, we do the same with free fields φ. Formulas for ϕ look the same as when definition 1. is extended to products of fields. Interaction term: $\varphi^{\star n}(x)$.

Perturbation theory on E_θ

- Graphs in $\mathbb{R}^3$: vertices, edges, labelled open edges
  $\#$ edges $+$ $\#$ free edges at a given vertex determined by P

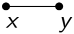
Perturbation theory on E_θ

- Graphs in $\mathbb{R}^3$: vertices, edges, labelled open edges
  # edges + # free edges at a given vertex determined by P
  Recall: there is only one propagator Δ_E. Only one way to do perturbation theory.

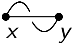
Perturbation theory on E_θ

- Graphs in $\mathbb{R}^3$: vertices, edges, labelled open edges
  # edges + # free edges at a given vertex determined by P
  Recall: there is only one propagator Δ_E. Only one way to do perturbation theory.
- Correspondence graphs $\leftrightarrow$ distributions [finite set of rules]:

$\leftrightarrow$ e^{ipx} $\quad p \in \mathbb{R}^4$ label



$\leftrightarrow$ $\Delta_E(x-y)$



$\leftrightarrow$ $\Delta_E(x-y) \star \Delta_E(x-y)$

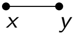
...

nonplanar graphs (crossing lines): twisted convolution products of Δ_E.

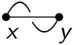
Perturbation theory on E_θ

- Graphs in $\mathbb{R}^3$: vertices, edges, labelled open edges
  $\#$ edges $+$ $\#$ free edges at a given vertex determined by P
  Recall: there is only one propagator Δ_E. Only one way to do perturbation theory.
- Correspondence graphs $\leftrightarrow$ distributions [finite set of rules]:

$\leftrightarrow \quad e^{ipx} \quad p \in \mathbb{R}^4$ label



$\leftrightarrow \quad \Delta_E(x-y)$



$\leftrightarrow \quad \Delta_E(x-y) \star \Delta_E(x-y)$

$\dots$

  nonplanar graphs (crossing lines): twisted convolution products of Δ_E.
- Rules comparatively simple (ribbon graphs).

Perturbation theory on M_θ

- There are different perturbative setups generalizing the ordinary rules [Doplicher+Fredenhagen, B+Doplicher+Fredenhagen+Piacitelli].
  No longer equivalent for maximal rank θ [B04].

Perturbation theory on M_θ

- There are different perturbative setups generalizing the ordinary rules [Doplicher+Fredenhagen, B+Doplicher+Fredenhagen+Piacitelli].
  No longer equivalent for maximal rank θ [B04].
- There are different ways to define the interaction term [B+Doplicher+Fredenhagen+Piacitelli].

Perturbation theory on M_θ

- There are different perturbative setups generalizing the ordinary rules [Doplicher+Fredenhagen, B+Doplicher+Fredenhagen+Piacitelli].
  No longer equivalent for maximal rank θ [B04].
- There are different ways to define the interaction term [B+Doplicher+Fredenhagen+Piacitelli].
- Fundamental open issues, e.g. Lorentz invariance.

Perturbation theory on M_θ

- There are different perturbative setups generalizing the ordinary rules [Doplicher+Fredenhagen, B+Doplicher+Fredenhagen+Piacitelli]. No longer equivalent for maximal rank θ [B04].
- There are different ways to define the interaction term [B+Doplicher+Fredenhagen+Piacitelli].
- Fundamental open issues, e.g. Lorentz invariance.
- This talk: approach which bears most similarity with noncommutative Euclidean Field Theory (cf. V. Rivasseau's talk): S-matrix with interaction term $\varphi^{\star n}$.

Perturbation theory on M_θ – continued

- S-matrix formalism. Described as an effective noncommutative field theory on $\mathbb{R}^4$, by replacing products of fields by twisted convolution products $\star$, i.e. with interaction Hamiltonian

$$H_I(t) = \int_{x^0=t} d^3\mathbf{x}\; g\,\varphi^{\star 3}(x)$$

Perturbation theory on M_θ – continued

- S-matrix formalism. Described as an effective noncommutative field theory on $\mathbb{R}^4$, by replacing products of fields by twisted convolution products $\star$, i.e. with interaction Hamiltonian

$$H_I(t) = \int_{x^0=t} d^3\mathbf{x}\; g\, \varphi^{\star 3}(x)$$

- Free theory and 2-point-function unchanged
  Consequence: distributions are the same as in QFT on $\mathbb{R}^4$:
  - Heaviside θ and 2-point-function Δ_+ on M_θ
  - (compare to Δ_E on E_θ).

Perturbation theory on M_θ – continued

- S-matrix formalism. Described as an effective noncommutative field theory on $\mathbb{R}^4$, by replacing products of fields by twisted convolution products $\star$, i.e. with interaction Hamiltonian

$$H_I(t) = \int_{x^0=t} d^3\mathbf{x}\; g\, \varphi^{\star 3}(x)$$

- Free theory and 2-point-function unchanged
  Consequence: distributions are the same as in QFT on $\mathbb{R}^4$:
  - Heaviside θ and 2-point-function Δ_+ on M_θ
  - (compare to Δ_E on E_θ).

  But: in general no Feynman propagators on M_θ.

A Glimpse at the Dyson series on M_θ

Example: $H_I(t) = \int_{x^0=t} d^3\mathbf{x}\; g\, \varphi^{\star 3}(x)$

A Glimpse at the Dyson series on M_θ

Example: $H_I(t) = \int_{x^0=t} d^3\mathbf{x}\; g\, \varphi^{\star 3}(x)$

$$S_2 \propto \int d^4x d^4y \,\left(\theta(x_0 - y_0)\; :\varphi(x)^{\star 3} : : \varphi(y)^{\star 3} : + (x \leftrightarrow y)\right) g(x) g(y)$$

A Glimpse at the Dyson series on M_θ

Example: $H_I(t) = \int_{x^0=t} d^3\mathbf{x}\; g\, \varphi^{\star 3}(x)$

$$S_2 \propto \int d^4x d^4y \left(\theta(x_0 - y_0)\ :\varphi(x)^{\star 3}::\varphi(y)^{\star 3}: + (x \leftrightarrow y)\right) g(x)g(y)$$

$$\langle p| :\varphi(x)^{\star 3}::\varphi(y)^{\star 3}: |q\rangle \ \rightarrow\ e^{-ipx}\, e^{iqy}\ \Delta_+^{\star 2}(x-y) + \ldots$$

A Glimpse at the Dyson series on M_θ

Example: $H_I(t) = \int_{x^0=t} d^3\mathbf{x}\; g\, \varphi^{\star 3}(x)$

$$S_2 \propto \int d^4x d^4y \left(\theta(x_0 - y_0)\; : \varphi(x)^{\star 3} : : \varphi(y)^{\star 3} : + (x \leftrightarrow y) \right) g(x) g(y)$$

$$\langle p| : \varphi(x)^{\star 3} : : \varphi(y)^{\star 3} : |q\rangle \;\rightarrow\; e^{-ipx}\, e^{iqy}\; \Delta_+^{\star 2}(x-y) + \ldots$$

$$\theta\; \Delta_+^{\star 2}(x-y) + \theta\; \Delta_+^{\star 2}(y-x) \;\neq\; \Delta_F^{\star 2}(x-y)$$

Perturbation theory on M_θ

- In nonplanar graphs (with crossing lines), time ordering (Heaviside function θ) and 2-point function Δ_+ can not in general be joined to yield (twisted convolution products of) Feynman propagators [B04].

Perturbation theory on M_θ

- In nonplanar graphs (with crossing lines), time ordering (Heaviside function θ) and 2-point function Δ_+ can not in general be joined to yield (twisted convolution products of) Feynman propagators [B04].
- If one naively puts Feynman propagators in nonplanar graphs $\Rightarrow$ violation of unitarity [Gomis+Minwalla 02].
  No problem with unitarity in careful analysis (based on effective Hamiltonian), where Heaviside and 2-point-function treated on a different footing [B, Doplicher, Fredenhagen, Piacitelli 03] as sketched above.

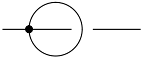
Perturbation theory on M_θ

- In nonplanar graphs (with crossing lines), time ordering (Heaviside function θ) and 2-point function Δ_+ can not in general be joined to yield (twisted convolution products of) Feynman propagators [B04].
- If one naively puts Feynman propagators in nonplanar graphs $\Rightarrow$ violation of unitarity [Gomis+Minwalla 02].
  No problem with unitarity in careful analysis (based on effective Hamiltonian), where Heaviside and 2-point-function treated on a different footing [B, Doplicher, Fredenhagen, Piacitelli 03] as sketched above.
- Price to pay: rules very complicated [B04, Piacitelli 04, Sibold 04, Denk+Schweda 04]. Not many calculations done so far.

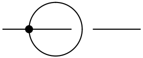
What is UV-IR mixing?

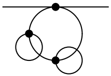
- Seiberg + Raamsdonck: The main tool of renormalization does not work on E_θ!

What is UV-IR mixing?

- Seiberg + Raamsdonck: The main tool of renormalization does not work on E_θ!
- How so?  is well defined

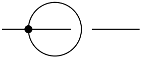
What is UV-IR mixing?

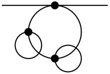
- Seiberg + Raamsdonck: The main tool of renormalization does not work on E_θ!
- How so?  is well defined
  but when inserted into higher order graph, e.g. (in $D = 4$)



these subgraphs suddenly turn ill defined!

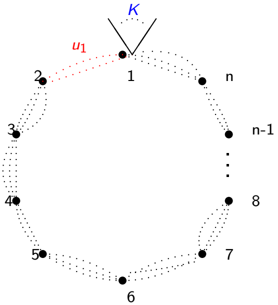
What is UV-IR mixing?

- Seiberg + Raamsdonck: The main tool of renormalization does not work on E_θ!
- How so?  is well defined
  but when inserted into higher order graph, e.g. (in $D = 4$)

  these subgraphs suddenly turn ill defined!
- Only way out: special models with translation invariance breaking term (harmonic oscillator) [Grosse+Wulkenhaar] $\rightarrow$ resummable? [Rivasseau]

Calculations were done in momentum space, actual mechanism not very well understood. Let's have another look (ordinary QFT):



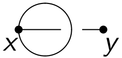
$$\int d^D x_1 \dots d^D x_n \; u_1(x_1 - x_2) \; u_2(x_2 - x_3) \cdots u_n(x_n - x_1) \; e^{iKx_1}$$

$$= \quad \delta(K) \int d^D p \; \tilde{u}_1(p) \; \tilde{u}_2(p) \cdots \tilde{u}_n(p)$$

The product of Fourier transforms of the distributions u_i appears!

On E_θ: exactly the same expression for the circle graph, different only in the distributions u that can appear,

e.g.



$\leftrightarrow \quad u_E(x-y)$

with

$$u_E(x-y) = \int d^D k d^D p \, \frac{1}{k^2+m^2} \, \frac{1}{p^2+m^2} \, e^{-ik\theta p} \, e^{ip(x-y)}$$

which is a C^∞-function if and only if twisting is present. In fact,

$$u_E(x-y) = \int d^D p \, \Delta_E(\theta p) \, \tilde{\Delta}_E(p) \, e^{ip(x-y)}$$

$\Rightarrow$ the Fourier transform of u_E contains both the Fourier transform $\tilde{\Delta}_E$ and Δ_E itself!

Origin of UV-IR mixing:

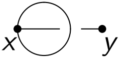
- In circle graph, products of distributions' Fourier transforms $\tilde{u}$ appear (via convolution of distributions u).

Origin of UV-IR mixing:

- In circle graph, products of distributions' Fourier transforms $\tilde{u}$ appear (via convolution of distributions u).
- For the distributions that usually appear in QFT (Δ_+, θ etc.) such products are well-defined distributions in momentum space.

Origin of UV-IR mixing:

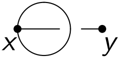
- In circle graph, products of distributions' Fourier transforms $\tilde{u}$ appear (via convolution of distributions u).
- For the distributions that usually appear in QFT (Δ_+, θ etc.) such products are well-defined distributions in momentum space.
- But products of the Fourier transform of



$= u_E(x - y)$ are not well-defined:

Origin of UV-IR mixing:

- In circle graph, products of distributions' Fourier transforms $\tilde{u}$ appear (via convolution of distributions u).
- For the distributions that usually appear in QFT (Δ_+, θ etc.) such products are well-defined distributions in momentum space.
- But products of the Fourier transform of

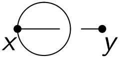
  

  $= u_E(x-y)$ are not well-defined:

  $\tilde{u}_E(p) = \Delta_E(\theta p)\tilde{\Delta}(p)$. In D dimensions, products Δ_E^n are ill-defined for $n \geq D-2$ due to well-known singularity in 0. Here: $p = 0$ (infrared).

Origin of UV-IR mixing:

- In circle graph, products of distributions' Fourier transforms $\tilde{u}$ appear (via convolution of distributions u).
- For the distributions that usually appear in QFT (Δ_+, θ etc.) such products are well-defined distributions in momentum space.
- But products of the Fourier transform of



$= u_E(x - y)$ are not well-defined:

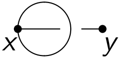
$\tilde{u}_E(p) = \Delta_E(\theta p)\tilde{\Delta}(p)$. In D dimensions, products Δ_E^n are ill-defined for $n \geq D - 2$ due to well-known singularity in 0. Here: $p = 0$ (infrared).

$\Rightarrow$ If u_E appears more than $(D - 2)$ times in a circle graph: divergence.

What is different on M_θ?

What is different on M_θ?

- Time ordering and 2-point-function come separately:



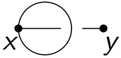
$$\leftrightarrow \quad u(x-y)$$

with

$$u(x-y) \quad = \quad \Theta(x_0 - y_0) \int d^D k d^D p \, \tilde{\Delta}_+(k) \, \tilde{\Delta}_+(p) \, e^{-ik\theta p} \, e^{ip(x-y)} \\ + (x \leftrightarrow y)$$

What is different on M_θ?

- Time ordering and 2-point-function come separately:



$$\leftrightarrow \quad u(x-y)$$

with

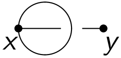
$$u(x-y) \;=\; \Theta(x_0-y_0)\int d^Dk\,d^Dp\;\tilde{\Delta}_+(k)\;\tilde{\Delta}_+(p)\;e^{-ik\theta p}\;e^{ip(x-y)} + (x\leftrightarrow y)$$

- u is a well-defined distribution if and only if twisting is present. Makes sense as oscillatory integral:

$$u(x-y) = \Theta(x_0-y_0)\int d^Dp\;\Delta_+(\theta p)\;\tilde{\Delta}_+(p)\;e^{ip(x-y)} + (x\leftrightarrow y)$$

What is different on M_θ?

- Time ordering and 2-point-function come separately:



$$\leftrightarrow \quad u(x-y)$$

with

$$u(x-y) \;=\; \Theta(x_0-y_0)\int d^Dk\,d^Dp\;\tilde{\Delta}_+(k)\;\tilde{\Delta}_+(p)\;e^{-ik\theta p}\;e^{ip(x-y)} + (x\leftrightarrow y)$$

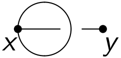
- u is a well-defined distribution if and only if twisting is present. Makes sense as oscillatory integral:

$$u(x-y)=\Theta(x_0-y_0)\int d^Dp\;\Delta_+(\theta p)\;\tilde{\Delta}_+(p)\;e^{ip(x-y)} + (x\leftrightarrow y)$$

- The Fourier transform of u contains both the Fourier transform $\tilde{\Delta}_+$ and Δ_+ itself.

Why no UV-IR mixing?

Products of the Fourier transform $\tilde{u}$ of



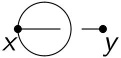
$= u(x-y)$ are well-defined:

$$\tilde{u}(p) = \int dk_0 \, \tilde{\Theta}(p_0 - k_0) \, \tilde{\Delta}_+(k_0, \mathbf{p}) \, \Delta_+(\theta(\omega_{\mathbf{p}}, \mathbf{p})) \; + \ldots$$

Time order Θ only affects $\tilde{\Delta}_+$, not tadpole part.

Why no UV-IR mixing?

Products of the Fourier transform $\tilde{u}$ of



$= u(x - y)$ are well-defined:

$$\tilde{u}(p) = \int d\,k_0\; \tilde{\Theta}(p_0 - k_0)\; \tilde{\Delta}_+(k_0, \mathbf{p})\; \Delta_+(\theta(\omega_{\mathbf{p}}, \mathbf{p}))\; + \ldots$$

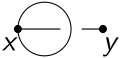
Time order Θ only affects $\tilde{\Delta}_+$, not tadpole part.

Theorem UV-IR mixing via convolutions with tadpole-like graphs is absent on M_θ [B07].

So far for φ^n for $n = 3, 4, 5, 6, 7$, since only a finite number of possible distributions has to be checked.

Why no UV-IR mixing?

Products of the Fourier transform $\tilde{u}$ of



$= u(x - y)$ are well-defined:

$$\tilde{u}(p) = \int d k_0 \, \tilde{\Theta}(p_0 - k_0) \, \tilde{\Delta}_+(k_0, \mathbf{p}) \, \Delta_+(\theta(\omega_{\mathbf{p}}, \mathbf{p})) \; + \ldots$$

Time order Θ only affects $\tilde{\Delta}_+$, not tadpole part.

Theorem UV-IR mixing via convolutions with tadpole-like graphs is absent on M_θ [B07].

So far for φ^n for $n = 3, 4, 5, 6, 7$, since only a finite number of possible distributions has to be checked.

Dispersion relation: IR behaviour is strange but (possibly) no hard mixing.

Outlook

- Exciting new possibilities: do full renormalization of quantum field theory in Dyson framework with interaction term $\varphi^{\star n}$ on M_θ. Investigate further whether UV-IR truly absent!

Outlook

- Exciting new possibilities: do full renormalization of quantum field theory in Dyson framework with interaction term $\varphi^{\star n}$ on M_θ. Investigate further whether UV-IR truly absent!

  What makes this so difficult?

Outlook

- Exciting new possibilities: do full renormalization of quantum field theory in Dyson framework with interaction term $\varphi^{\star n}$ on M_θ. Investigate further whether UV-IR truly absent!

  What makes this so difficult?

  - No cyclic symmetry,
    only 3-momentum conservation at vertex.
  - Often graphs only make sense as oscillatory integrals, whereas in Euclidean framework, often finite graphs correspond to C^∞-functions.
  - Mixture of position/momentum space renormalization methods needed.
  - Lack of Feynman propagators $\Rightarrow$ only limited use of known techniques.

Outlook – continued

- Complementary step in the direction of general renormalization proof combinatorics of quasiplanar Wick products [BDFP] understood in terms of shuffle Hopf algebra on chord diagrams [B07].

Outlook – continued

- Complementary step in the direction of general renormalization proof combinatorics of quasiplanar Wick products [BDFP] understood in terms of shuffle Hopf algebra on chord diagrams [B07].
- Test other models (non-central commutator, κ-deformed...) for UV/IR properties – is this special for M_θ?

Outlook – continued

- Complementary step in the direction of general renormalization proof combinatorics of quasiplanar Wick products [BDFP] understood in terms of shuffle Hopf algebra on chord diagrams [B07].
- Test other models (non-central commutator, κ-deformed...) for UV/IR properties – is this special for M_θ?
- Is there some analytic continuation of nc Lorentzian field theory? What sort of Euclidean field theory could it yield?

Outlook – continued

- Complementary step in the direction of general renormalization proof combinatorics of quasiplanar Wick products [BDFP] understood in terms of shuffle Hopf algebra on chord diagrams [B07].
- Test other models (non-central commutator, κ-deformed...) for UV/IR properties – is this special for M_θ?
- Is there some analytic continuation of nc Lorentzian field theory? What sort of Euclidean field theory could it yield?
- Euclidean and Lorentz field theory shown to be quite different from one another. Look again at low energy limits in string theory in Lorentzian framework.

Outlook – continued

- Complementary step in the direction of general renormalization proof combinatorics of quasiplanar Wick products [BDFP] understood in terms of shuffle Hopf algebra on chord diagrams [B07].
- Test other models (non-central commutator, κ-deformed...) for UV/IR properties – is this special for M_θ?
- Is there some analytic continuation of nc Lorentzian field theory? What sort of Euclidean field theory could it yield?
- Euclidean and Lorentz field theory shown to be quite different from one another. Look again at low energy limits in string theory in Lorentzian framework.
- Is there an analytic continuation of the Grosse-Wulkenhaar model to M_θ?

Outlook (speculation)

- On a more general level: understand Lorentz invariance violation... Gauge theories: where is the S of $SU(n)$?...
- Possibly necessary: think about noncommutative space as a space of internal degrees of freedom – as in Connes' standard model!

  Framework [?]: tensor products of fields defined for quasiplanar Wick products [BDFP]:

  $$\int dx_1 \dots dx_n \; f(x_1, \dots, x_n) \; \phi(q + x_1) \dots \phi(q + x_n)$$

  Rethink: not $x_i \in \mathbb{R}^4$ are auxilliary, but the noncommuting coordinates q!
- Connection with work by Grosse and Lechner? Buchholz and Lechner?