# Quantum field theory on projective modules

T. Krajewski

Centre de Physique Théorique (Marseille)
krajew@cpt.univ-mrs.fr

Joint work with V. Gayral, J.-H. Jureit and R. Wulkenhaar
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*"Commutative and noncommutative quantum fields"*
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**Plan of the talk:**

- Projective modules over the noncommutative torus
- Action functional
- Feynman diagrams
- Matrix model
- Renormalization

# The noncommutative torus

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The **differential calculus** is constructed using the **derivations** $\partial_\mu U_\gamma = 2\mathrm{i}\pi\gamma_\mu \, U_\gamma$ and the **trace** $\mathrm{Tr}_{\mathcal{A}_\theta}(U_\gamma) = \delta_{\gamma,0}$ that plays the role of an integral.

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The **Heisenberg representation** on functions on $\mathbb{R}$ defines a projective module $\mathcal{E}$ over $\mathcal{A}_\theta$

$$\big(\phi U_\gamma\big)(x) = \mathrm{e}^{\mathrm{i}\pi\theta mn}\,\mathrm{e}^{2\mathrm{i}\pi nx}\,\phi(x+m\theta)$$

with a **connection** $\nabla_\mu : \mathcal{E} \to \mathcal{E}$

$$\nabla_1\phi(x) = -\frac{2i\pi x}{\theta}\phi(x) \quad \text{and} \quad \nabla_2\phi(x) = \frac{d\phi(x)}{dx}.$$

compatible with the **hermitian structure** $\mathcal{E}\times\mathcal{E}\to\mathcal{A}_\theta$

$$(\phi,\chi)_{\mathcal{A}_\theta} = \sum_{\gamma\in\Gamma}\left(\mathrm{e}^{i\pi\theta mn}\int_{\mathbb{R}} dx\,\overline{\phi}(x)\,\mathrm{e}^{2i\pi nx}\,\chi(x{+}m\theta)\right)\,U_{-\gamma}$$

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This projective module corresponds to the wave functions of the **lowest Landau level** and the action of $\mathcal{A}_\theta$ encodes the **magnetic translations** along a lattice with $\theta$ the magnetic flux through the unit cell measured in terms of the flux quantum $\frac{\hbar}{e}$.

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- The kinetic term is identical to the **Harmonic oscillator**
- The action is invariant under the **Langmann-Szabo duality**

$$S_{\lambda,\,\mu,\,\theta}[\phi,\overline{\phi}] = \tfrac{1}{\theta^2}\,S_{\theta\lambda,\,\theta^2\mu,\,1/\theta}[\eta,\overline{\eta}]$$

with $\eta(\xi) = \int_{\mathbb{R}} dx\,e^{-2i\pi x\xi}\,\phi(x)$ the Fourier transform of $\phi$

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To compute the **correlation functions**

$$G_{2N} = \frac{\displaystyle\int_{\mathcal{E}} [D\phi][D\phi^\dagger]\ (\phi \otimes \phi^\dagger) \cdots (\phi \otimes \phi^\dagger)\ \mathrm{e}^{-S[\phi,\phi^\dagger]}}{\displaystyle\int_{\mathcal{E}} [D\phi][D\phi^\dagger]\ \mathrm{e}^{-S[\phi,\phi^\dagger]}}$$

we introduce the **Hubbard-Stratonovitch** auxiliary field

$$\mathrm{e}^{-\frac{\lambda}{2}\mathrm{Tr}_{\mathcal{A}_\theta}\left[(\phi,\phi)^2_{\mathcal{A}_\theta}\right]} = \int_{\mathcal{A}_\theta} [DA]\ \mathrm{e}^{-\mathrm{Tr}_{\mathcal{A}_\theta}\left[\frac{\lambda}{2}A^2 + \mathrm{i}\lambda A(\phi,\phi)_{\mathcal{A}_\theta}\right]}$$

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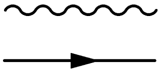

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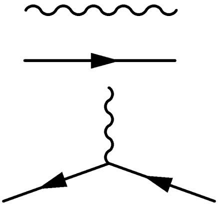
- Interacting vertex between $\phi$ and $A$ — $i\lambda U_\gamma$



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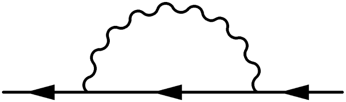

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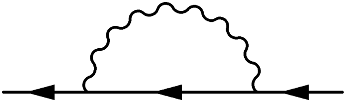

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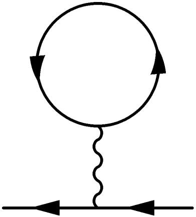

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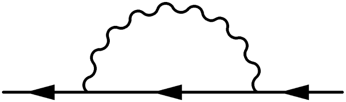

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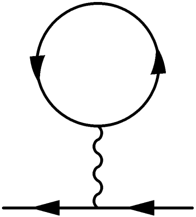

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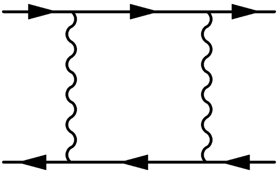

$$\lambda^2\sum_{\gamma_1,\gamma_2}\left(U_{-\gamma_1}\, H^{-1}\, U_{-\gamma_2}\right)\otimes\left(U_{\gamma_2}\, H^{-1}\, U_{\gamma_1}\right)$$

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(1)

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For $\theta = p/q$ the projective module can be realized as a **bundle** of **rectangular** $p \times q$ matrices on a torus of size $1/q$

$$M_{ij}(x, y) = \sum_{n \in \mathbb{Z}} \phi\Big(x + \frac{iq + jp + npq}{q}\Big) \mathrm{e}^{-2\mathrm{i}\pi nqy} \tag{1}$$

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with the **twisted boundary conditions**

$$\begin{cases} M(x, y + \frac{1}{q}) = M(x,y) \\ M(x + \frac{1}{q}, y) = \Omega_p^a(qy) M(x,y) \Omega_q^{-b}(-qy) \end{cases}$$

where $a$ and $b$ are two integers such that $aq + bp = 1$ and $\Omega_N(y)$ is the $N \times N$ matrix defined by

$$\Omega_N(y) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ e^{2i\pi y} & & & 0 \end{pmatrix} \qquad (1)$$

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Since the action on the projective module only depends on $p/q$ it is possible to take the limit of **large matrices**

$$\lim_{\substack{p,q\to\infty \\ p/q\to\theta}} \int [DM][DM^\dagger] \mathrm{e}^{-qS[M,M^\dagger]} \cdots = \int [D\phi][D\overline{\phi}] \mathrm{e}^{-S[\phi,\overline{\phi}]} \ldots$$

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As we take the limit of large matrices, the torus shrinks to a **single point**.

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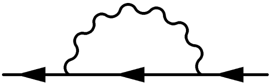

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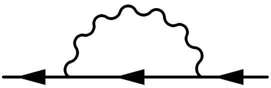

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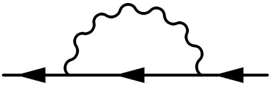

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Sums over $\gamma$ are convergent (as distributions) as soon as $\alpha \neq 0$.

The theory is **renormalizable** for irrational $\theta$ satisfying a **Diophantine condition** provided we add a new counternterm

$$\lambda' \left[ \mathrm{Tr}_{\mathcal{A}_\theta} \left( \phi, \phi \right)_{\mathcal{A}_\theta} \right]^2$$

# Planar vs non-planar at $\alpha = 0$

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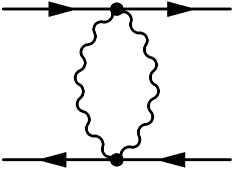
**Planar** diagrams have an **ordinary** divergence



$$\sum_{\gamma_1,\gamma_2} U_{\gamma_1} U_{\gamma_2} \otimes U_{-\gamma_1} U_{-\gamma_2} = \sum_{\gamma=\gamma_1+\gamma_2} U_{\gamma} \otimes U_{-\gamma} \left( \sum_{\gamma_1} 1 \right)$$

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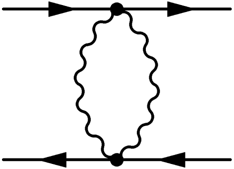

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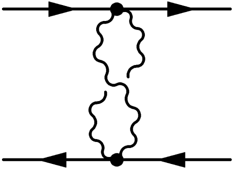
**Non-planar** diagrams exhibit a special **divergence** at $\gamma = 0$



$$\sum_{\gamma_1,\gamma_2} U_{\gamma_1} U_{\gamma_2} \otimes U_{-\gamma_2} U_{-\gamma_1} = \sum_{\gamma=\gamma_1+\gamma_2} U_{\gamma} \otimes U_{-\gamma} \left( \sum_{\gamma_1} e^{2i\pi\theta(\gamma_1,\gamma)} \right)$$