Spectral Action on noncommutative torus

Cyril LEVY

D. ESSOUABRI, B. IOCHUM, & A. SITARZ : ArXiv 2007 - JNCG to appear

Vienna, November 30, 2007

Why spectral triples ? Physical motivation

Geometrization of particle physics

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{EH}$$

M : 4-dimensional manifold (spacetime)

- Symmetries of $\mathcal{L}_{EH}$: $\mathbf{Diff(M)}$
- Symmetries of $\mathcal{L}_{SM}$: $\mathcal{G}_{\mathbf{SM}}$
- Full lagrangian : $\mathcal{G}_{\mathbf{SM}} \rtimes \mathbf{Diff(M)}$

Why spectral triples ? Physical motivation

Geometrization of particle physics

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{EH}$$

M : 4-dimensional manifold (spacetime)

- Symmetries of $\mathcal{L}_{EH}$: $\mathbf{Diff(M)}$
- Symmetries of $\mathcal{L}_{SM}$: $\mathcal{G}_{\mathbf{SM}}$
- Full lagrangian : $\mathcal{G}_{\mathbf{SM}} \rtimes \mathbf{Diff(M)}$

Question : is there X such that $Diff(X) = \mathcal{G}_{SM} \rtimes Diff(M)$?

Why spectral triples ? Physical motivation

Geometrization of particle physics

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{EH}$$

M : 4-dimensional manifold (spacetime)

• Symmetries of $\mathcal{L}_{EH}$: $\mathbf{Diff(M)}$
• Symmetries of $\mathcal{L}_{SM}$: $\mathcal{G}_{\mathbf{SM}}$
• Full lagrangian : $\mathcal{G}_{\mathbf{SM}} \rtimes \mathbf{Diff(M)}$

Question : is there X such that $Diff(X) = \mathcal{G}_{SM} \rtimes Diff(M)$?

Answer : No, But...

Noncommutative generalization of manifolds (*Spectral Triples*) do the job.

Spectral triples

- Commutative world :

$$(C^\infty(M), L^2(M,S), \mathcal{D})$$

  - operators $[\mathcal{D}, f]$ bounded
  - if $n = \dim M$, $|\mathcal{D}|^{-n}$ infinitesimal of degree 1.
  - $\mathcal{D} \simeq$ metric, $\mathcal{D}^{-1} \simeq ds$

$$d(x,y) = \sup\{|f(x) - f(y)| : f \in C^\infty(M), \|[\mathcal{D}, f]\| \leq 1\}$$

Spectral triples

- Commutative world :

$$(C^\infty(M), L^2(M,S), \mathcal{D})$$

  - operators $[\mathcal{D}, f]$ bounded
  - if $n = \dim M$, $|\mathcal{D}|^{-n}$ infinitesimal of degree 1.
  - $\mathcal{D} \simeq$ metric, $\mathcal{D}^{-1} \simeq ds$

$$d(x,y) = \sup\{|f(x) - f(y)| : f \in C^\infty(M), \|[\mathcal{D}, f]\| \leq 1\}$$

- Noncommutative world :

$$(\mathcal{A}, \mathcal{H}, \mathcal{D})$$

$+J$: real structure

Real structure and Inner fluctuation

Charge conjugaison operator for QFT : J

$$J\mathcal{D} = \varepsilon \mathcal{D} J \qquad J^2 = \varepsilon' \qquad J\chi = \varepsilon'' \chi J$$

- $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ depends on $n \mod 8$.

Gauge potential ? Fluctuation of the metric.

- Morita equivalence $\mathcal{A} \simeq End_{\mathcal{A}}(\mathcal{A})$ gives

$$\mathcal{D} \to \mathcal{D} + A \qquad A = \sum_i a_i [\mathcal{D}, b_i]$$

- Real Structure :

$$\mathcal{D} \to \mathcal{D}_A := \mathcal{D} + A + \varepsilon J A J^{-1}$$

- $A =$ gauge potential as inner fluctuation of $\mathcal{D}$.

The spectral action principle

• What can play the role of the action in $(\mathcal{A}, \mathcal{H}, \mathcal{D})$?

The spectral action principle

• What can play the role of the action in $(\mathcal{A}, \mathcal{H}, \mathcal{D})$?

$$\mathcal{S}(\mathcal{D}, \Phi, \Lambda) = \mathrm{Tr}\big(\Phi(\mathcal{D}/\Lambda)\big)$$

• Φ = cut-off function
• Λ = mass scale
• Action based on "spectral data" : $\mathcal{S}$ counts the number of eigenvalues of $|\mathcal{D}| < \Lambda$.

Computation of $\mathcal{S}(\mathcal{D}, \Phi, \Lambda)$

One approach : Heat Kernel expansion and Zeta functions.

$$\mathrm{Tr}(e^{-t\mathcal{D}^2}) \sim \sum_\alpha a_\alpha t^\alpha \qquad (t \to 0)$$

- $\mathrm{Res}_{s=-2\alpha}\zeta_\mathcal{D}(s) = c_\alpha\, a_\alpha$
- $\zeta_\mathcal{D}(0) = a_0$
- $\zeta_\mathcal{D}(s) = \mathrm{Tr}(|\mathcal{D}|^{-s})$
- $⨍ T = \mathrm{Res}_{s=0}\mathrm{Tr}(T|\mathcal{D}|^{-s})$

Computation of $\mathcal{S}(\mathcal{D}, \Phi, \Lambda)$

One approach : Heat Kernel expansion and Zeta functions.

$$\mathrm{Tr}(e^{-t\mathcal{D}^2}) \sim \sum_\alpha a_\alpha t^\alpha \qquad (t \to 0)$$

- $\mathrm{Res}_{s=-2\alpha} \zeta_{\mathcal{D}}(s) = c_\alpha \, a_\alpha$
- $\zeta_{\mathcal{D}}(0) = a_0$
- $\zeta_{\mathcal{D}}(s) = \mathrm{Tr}(|\mathcal{D}|^{-s})$
- $⨍ T = \mathrm{Res}_{s=0} \mathrm{Tr}(T|\mathcal{D}|^{-s})$

Laplace transform and $t = \Lambda^{-1}$:

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0<k\in Sd^+} \Phi_k \, \Lambda^k ⨍ |\mathcal{D}_A|^{-k} + \Phi(0)\, \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1})$$

- Sd : Spectrum dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Coefficients of the spectral action

Computation of $\zeta_{D_A}(0)$ and $⨍ |\mathcal{D}_A|^{-k}$: ΨDO calculus needed.

$$OP^0 = \bigcap_{n\in\mathbb{N}} \text{Dom } \delta^n \qquad \delta(T) := [|\mathcal{D}|, T] \qquad \mathcal{A},\ J\mathcal{A}J^{-1},\ [\mathcal{D},\mathcal{A}] \subset OP^0$$

$$OP^\alpha = \{\, T : T|\mathcal{D}|^{-\alpha} \in OP^0 \,\}, \qquad OP^\alpha OP^\beta \subset OP^{\alpha+\beta}, \qquad OP^{-n-\varepsilon} \subset \mathcal{L}^1(\mathcal{H})$$

$T \in \Psi DO^d$ if $\forall N \in \mathbb{N}$

$$T = P_N D^{-2p_N} + R_N$$

$$p_N \in \mathbb{N} \qquad P_N \in \text{Alg}\{\, \mathcal{A}, J\mathcal{A}J^{-1}, \mathcal{D}, |\mathcal{D}| \,\}$$

$$R_N \in OP^{-N} \qquad P_N|\mathcal{D}|^{-2p_N} \in OP^d$$

Spectrum dimension : poles of $\zeta_{\mathcal{D}}^T(s) = \mathrm{Tr}(T|\mathcal{D}|^{-s}),\ T \in OP^0$.

Commutative case : $Sd = \{n - k \ :\ k \in \mathbb{N}\}$.

Coefficients of the spectral action

Simple spectrum dimension $\Longrightarrow ⨍$ trace on the algebra ΨDO : $⨍ PQ = ⨍ QP$.

$|\mathcal{D}_A|^{-k}$ are pseudodifferential operators and

$$⨍ |\mathcal{D}_A|^{-k} = ⨍ |\mathcal{D}|^{-k} + \sum_{p=1}^{n-k} \sum_{r_1,\cdots,r_p=0}^{n-k-p} h(k,r,p) ⨍ A(r_1,\cdots,r_p)|\mathcal{D}|^{-k},$$

Scale-invariant term (Chamseddine-Connes 2006)

$$\zeta_{\mathcal{D}_A}(0) = \sum_{q=1}^{n} \frac{(-1)^q}{q} ⨍ ((A + JAJ^{-1})|\mathcal{D}|^{-1})^q$$

Coefficients computation $\rightarrow$ residues of $\mathcal{D}$-zeta functions :

$$\mathrm{Res}_{s=p}\ \zeta_{\mathcal{D}}^{P}(s) = \mathrm{Res}_{s=p}\ \mathrm{Tr}(P|\mathcal{D}|^{-s}) \qquad P \in \Psi DO$$

NC-torus of dimension d

$$\mathcal{A} := \{\, a = \sum_{k\in\mathbb{Z}^d} a_k\, U_k \;:\; (a_k)_k \in \mathcal{S}(\mathbb{Z}^d)\,\}$$

$$U_k\, U_q = e^{-ik.\Theta q}\, U_q\, U_k$$

Θ skew symmetric $d \times d$-matrix.

U_k are unitaries
$\tau(a) := a_0$ is a trace giving by GNS $\mathcal{H}_\tau$

$$\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}, \quad m = [d/2]$$

Natural derivations : $\delta_\mu, \quad \mu \in \{\, 1, \dots, d \,\}$

$$\delta_\mu U_k := ik_\mu\, U_k$$

Dirac operator

$$\mathcal{D} := -i\delta_\mu \otimes \gamma^\mu$$

NC-torus of dimension d

$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple

Hermitian one-form $A := \sum_i a_i[\mathcal{D}, b_i],\ a_i,\ b_i \in \mathcal{A}$

$$\begin{aligned}\mathcal{D}_A &= \mathcal{D} + A + \epsilon JAJ^{-1} \\ &= -i\big(\delta_\mu + L(A_\mu) - R(A_\mu)\big) \otimes \gamma^\mu, \quad A_\mu = -{A_\mu}^*\end{aligned}$$

• Result : Under a number theoretical hypothesis on Θ,
Sd(Nc-torus)$= \{\, d - k \ :\ k \in \mathbb{N}_0 \,\}$ and all poles are simple (simple spectrum dimension).

• Whole spectral action reduced to computation of residues of zeta functions

• NC-torus $\rightarrow$ (Schwartz weighted series of) Hurwitz-Epstein zeta functions.

Residues of some Hurwitz-Epstein zeta functions

Result :

$$f_a(s) := \sum_{0 \neq k \in \mathbb{Z}^d} \frac{P(k)}{||k||^s} \, e^{i2\pi \, k.a}$$

$a \in \mathbb{R}^d$, $P \in \mathbb{C}[x_1, \ldots, x_d]$ homogeneous polynomial of degree p,
$||k|| = \sqrt{{k_1}^2 + \ldots {k_d}^2}$.
- When $a \in \mathbb{Z}^d$, f_a has meromorphic extension to $\mathbb{C}$.
f_a not entire $\Longleftrightarrow$ $\text{Res}_{s=d+p} \, f_a(s) = \int_{u \in S^{d-1}} P(u) \, dS(u) \neq 0$
- When $a \in \mathbb{R}^d \setminus \mathbb{Z}^d$, f_a has an analytic extension to $\mathbb{C}$.

Residues of some Hurwitz-Epstein zeta functions

Result :

$$f_a(s) := \sum_{0 \neq k \in \mathbb{Z}^d} \frac{P(k)}{||k||^s} \, e^{i2\pi \, k.a}$$

$a \in \mathbb{R}^d$, $P \in \mathbb{C}[x_1, \dots, x_d]$ homogeneous polynomial of degree p,
$||k|| = \sqrt{{k_1}^2 + \dots {k_d}^2}$.
- When $a \in \mathbb{Z}^d$, f_a has meromorphic extension to $\mathbb{C}$.
f_a not entire $\Longleftrightarrow$ $\mathrm{Res}_{s=d+p} \, f_a(s) = \int_{u \in S^{d-1}} P(u) \, dS(u) \neq 0$
- When $a \in \mathbb{R}^d \setminus \mathbb{Z}^d$, f_a has an analytic extension to $\mathbb{C}$.
Schwartz weighted series of f_a :
When Θ is badly approximable, for any $q > 0$,

$$g(s) := \sum_{l \in (\mathbb{Z}^d)^q} c(l) \, f_{\Theta(\sum_i l_i)}(s), \quad c(l) \in \mathcal{S}(\mathbb{Z}^d)^q$$

has only one pole on $s = d + p$ with

$$\mathrm{Res}_{s=d+p} \, g(s) = c \int_{u \in S^{d-1}} P(u) \, dS(u)$$

Diophantine Approximation

Θ badly approximable :

Diophantine condition : $\exists u \in \mathbb{Z}^d$, $\exists \delta > 0$

$$|q.\Theta u - p| > c|q|^{-\delta}, \quad \forall 0 \neq q \in \mathbb{Z}^d, \quad \forall p \in \mathbb{Z}$$

Almost any vector in $\mathbb{R}^n$ is badly approximable.

Due to J : Control holomorphy of Schwartz weighted Hurwitz-Epstein Zeta functions with Θ-phase

NC-torus of dimension d : Action

Result

$d = 2$:

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \phi) = 4\pi\, \phi_2\, \Lambda^2 + \mathcal{O}(\Lambda^{-2})$$

$d = 3$:

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \phi) = 8\pi\, \phi_3\, \Lambda^3 + \mathcal{O}(\Lambda^{-1})$$

$d = 4$:

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \phi) = 8\pi^2\, \phi_4\, \Lambda^4 - \tfrac{4\pi^2}{3}\, \tau(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2})$$

$F_{\mu\nu} = \delta_\mu(A_\nu) - \delta_\nu(A_\mu) - [A_\mu, A_\nu]$.

More generally, $\forall d \geq 1$,

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \phi) = \sum_{k=0}^{d} \phi_{d-k}\, c_{d-k}(A)\, \Lambda^{d-k} + \mathcal{O}(\Lambda^{-1})$$

$c_{d-2}(A) = 0,\ c_{d-k}(A) = 0$ for k odd (d odd $\Rightarrow c_0(A) = 0$)

Conjecture : Noncommutative coefficients of $\mathcal{D} + \widetilde{A} \simeq$ coefficients of $\mathcal{D} + A$ for the commutative torus

Final remarks

- No tadpole on nc-torus : $\fint \widetilde{A}\mathcal{D}^{-1} = 0$

- Top term (cosmological term) : $\fint |\mathcal{D}_A|^{-d} = \fint |\mathcal{D}|^{-d}$

- Non invertibility of $\mathcal{D}$ and $\mathcal{D}_A$: not a problem because P_0 and P_A are in $OP^{-\infty}$.

- Open question : Is the badly approximable hypothesis on Θ necessery, can it be weakened ?

- Spectral action on Noncompact spectral triple ?