Noncommutative Gravity

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noncommutative spacetime
introduction + motivation

Noncommutative / Quantum Spacetime

picture: Marianne Freiberger, +plus

Noncommutative / Quantum Spacetime

- Spacetime manifold is replaced by a noncommutative algebra of „functions on NC spacetime“

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Noncommutative / Quantum Spacetime

- Priciples of quantum mechanics are applied to spacetime itself

$$[x^i, x^j] = \theta^{ij}(x)$$

- Spacetime manifold is replaced by a noncommutative algebra of „functions on NC spacetime“

picture: Marianne Freiberger, +plus

Noncommutative / Quantum Spacetime

- Model of short distance structure of spacetime
  $\rightarrow$ uncertainty relation for position measurements

- Priciples of quantum mechanics are applied to spacetime itself
  $$[x^i,x^j] = \theta^{ij}(x)$$

- Spacetime manifold is replaced by a noncommutative algebra of „functions on NC spacetime“

picture: Marianne Freiberger, +plus

PHYSICAL REVIEW VOLUME 71, NUMBER 1 JANUARY 1, 1947

Quantized Space-Time

HARTLAND S. SNYDER
Department of Physics, Northwestern University, Evanston, Illinois
(Received May 13, 1946)

It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time.

THE problem of the interaction of matter and fields has not been satisfactorily solved to this date. The root of the trouble in present field theories seems to lie in the assumption of point interactions between matter and fields. On the other hand, no relativistically invariant Hamiltonian theory is known for any form of interaction other than point interactions.

Even for the case of point interactions the relativistic invariance is of a formal nature only, as the equations for quantized interacting fields have no solutions. The uses of source functions, or of a cut-off in momentum space to replace infinity by a finite number are distasteful arbitrary procedures, and neither process has yet been formulated in a relativistically invariant manner. It may not be possible to do this.

It is possible that the usual four-dimensional continuous space-time does not provide a suitable framework within which interacting matter and fields can be described. I have chosen the idea that a modification of the ordinary concept of space-time may be necessary because the "elementary" particles have fixed masses and associated Compton wave-lengths.

The special theory of relativity may be based on the invariance of the indefinite quadratic form

$$S^2 = c^2t^2 - x^2 - y^2 - z^2, \tag{1}$$

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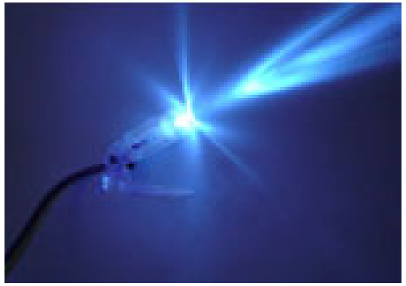
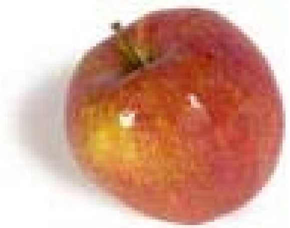
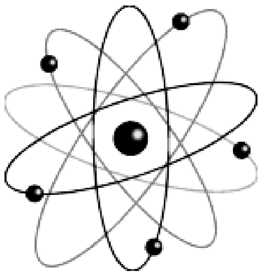


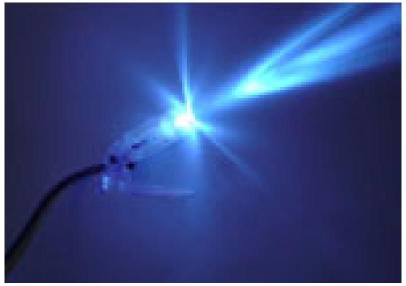
c

position measurement with precision Δ x, requires...

G 

$\hbar$





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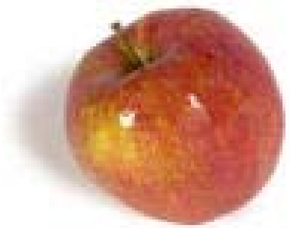

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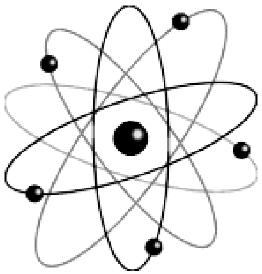
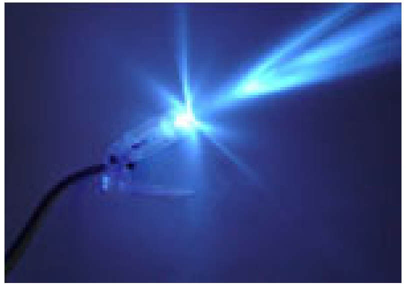


momentum $\hbar$ / Δ x



energy $\hbar c$ / Δ x c

position measurement with precision Δ x, requires...

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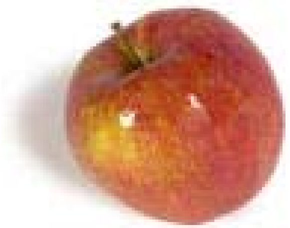

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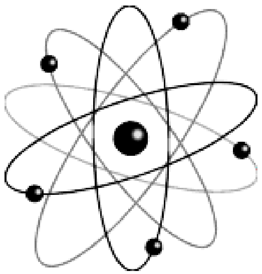

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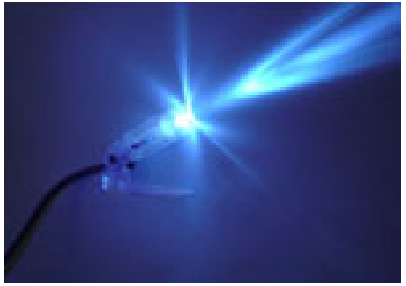
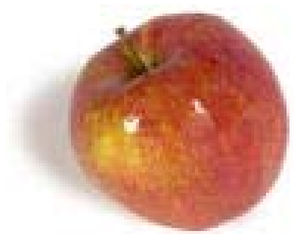


energy $\hbar c$ / Δ x $\quad c$

creates horizon of size $G\hbar^{-3}$ / Δx

position measurement with precision Δ x, requires...

G



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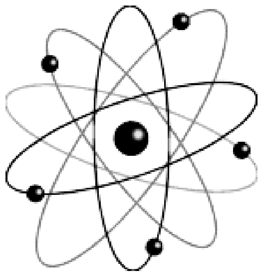
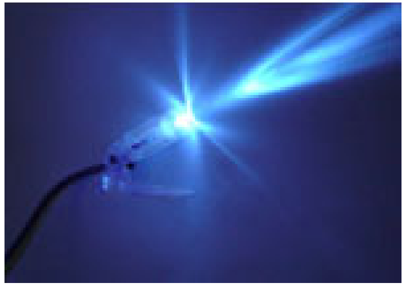


momentum $\hbar$ / Δ x

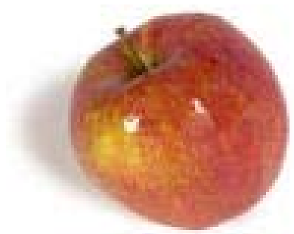


c

If horizon $G\hbar\, c^{-3}/\Delta x > \Delta x \Rightarrow$ information is lost

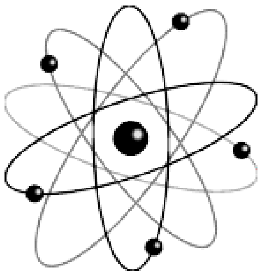
$$\Rightarrow \Delta x \geq \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}\text{cm}$$

G



spacetime uncertainty

$\hbar$



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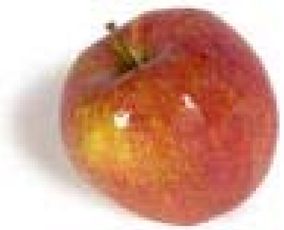
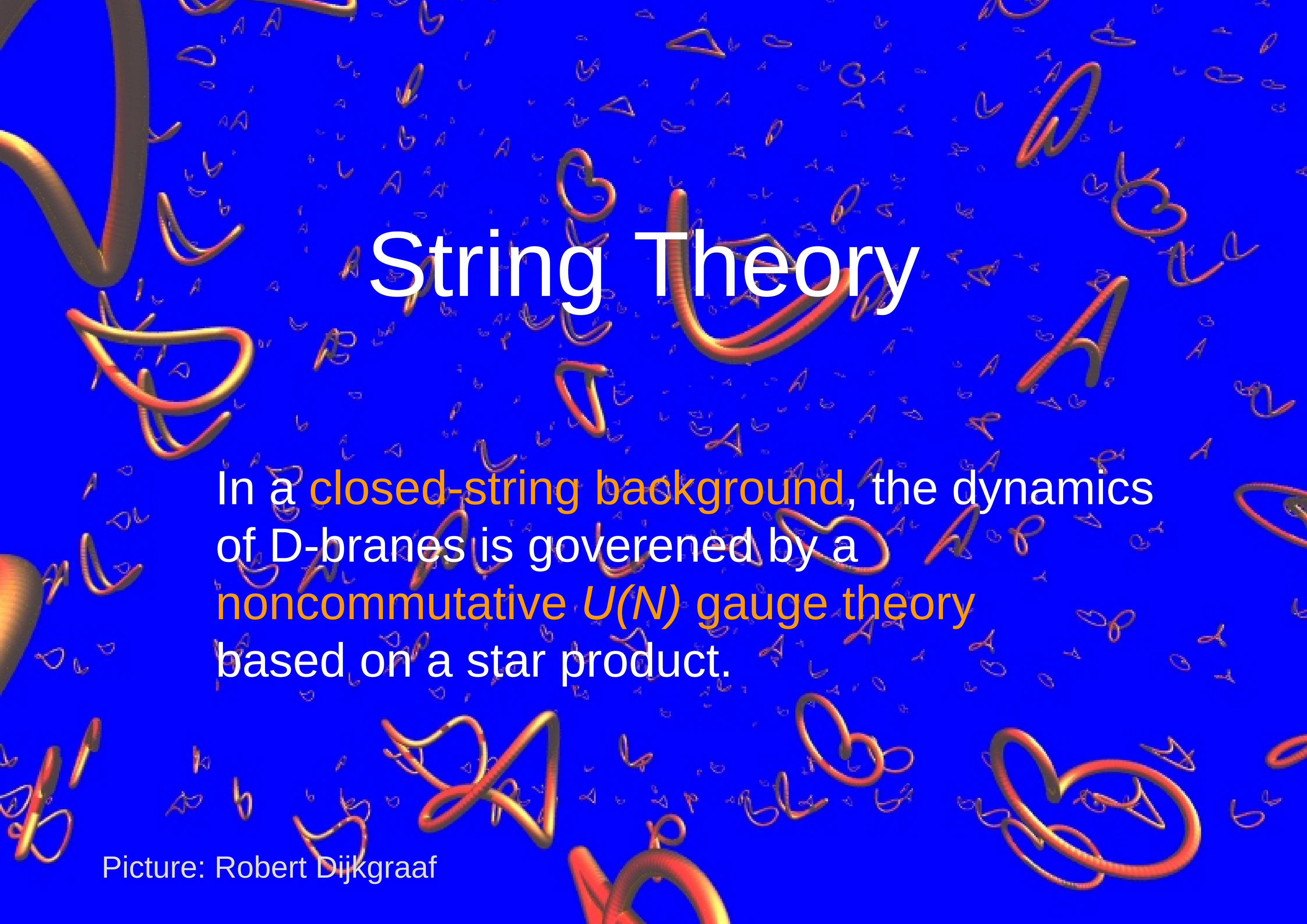


noncommutative spacetime

$$[\widehat{x}^i, \widehat{x}^j] = i\theta^{ij}$$

String Theory

In a closed-string background, the dynamics of D-branes is goverened by a noncommutative *U(N)* gauge theory based on a star product.



Picture: Robert Dijkgraaf

Star product

- Noncommutative associative algebra realized on ordinary functions via a bi-differential operator

$$f \star g = f \cdot g + \frac{ih}{2}\theta^{ij}\partial_i f \cdot \partial_j g + \ldots$$

- „Spacetime“ ($t, x_1, x_2, x_3, \ldots$) becomes an auxilliary space

- Efficient way to implement field theory on NC spaces

Field theory on NC spacetime

Field theory on NC spacetime

Functions on spacetime; coordinates, as well as classical fields are promoted to operators

$$\phi \mapsto \widehat{\phi}$$

(„functions on NC spacetime“)

Spacetime noncommutativity is often formulated via star products

$$\widehat{\phi}\,\widehat{\psi} =: \widehat{\phi \star \psi}$$

Field theory on NC spacetime

Classical fields are still ordinary functions, but products of fields are replaced by star products

$$f \star g = fg + \frac{i\hbar}{2} \sum_{i,j} \theta^{ij}\, \partial_i(f)\, \partial_j(g) + \frac{-\hbar^2}{4} \sum_{i,j,k,l} \theta^{ij}\theta^{kl}\, \partial_i\partial_k(f)\, \partial_j\partial_l(g) +$$

$$+ \frac{-\hbar^2}{6} \left(\sum_{i,j,k,l} \theta^{ij}\, \partial_j(\theta^{kl})\, (\partial_i\partial_k(f)\, \partial_l(g) - \partial_k(f)\, \partial_i\partial_l(g)) \right) + O(\hbar^3)$$

Field theory on NC spacetime

There is a huge “gauge“ symmetry:

Any invertible formal differential operator *D* yields a new equivalent star product

$$D(f \star' g) = Df \star Dg$$

Field theory on NC spacetime

The simplest, most popular and furthermore translationally symmetric example...

Moyal-Weyl

$$f\star g = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\,\partial_{\mu_1}\ldots\partial_{\mu_n} f\cdot\partial_{\nu_1}\ldots\partial_{\nu_n} g$$

$$x^\mu\star x^\nu = x^\mu x^\nu + \frac{i\hbar}{2}\theta^{\mu\nu}\,, \qquad [x^\mu \overset{\star}{,} x^\nu] = i\hbar\theta^{\mu\nu}$$

Field theory on NC spacetime

... arises from an abelian twist:

$$\mathcal{F} = \exp\left(-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\right)$$

$$f \star g = \mu(\mathcal{F}^{-1}(f \otimes g))$$

Field theory on NC spacetime

... it can be generalized:

$$\mathcal{F} = \exp\left(-\frac{i}{2}\theta^{ab}\, V_a \otimes V_b\right) \;, \qquad\qquad [V_a, V_b] = 0$$

$$f \star g = \mu(\mathcal{F}^{-1}(f \otimes g))$$

Field theory on NC spacetime

... and it fixes two directions in spacetime that badly break global as well as local spacetime symmetry.

$$f \star g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar}{2} \right)^n \theta^{\mu_1 \nu_1} \ldots \theta^{\mu_n \nu_n} \, \partial_{\mu_1} \ldots \partial_{\mu_n} f \cdot \partial_{\nu_1} \ldots \partial_{\nu_n} g$$

$$x^\mu \star x^\nu = x^\mu x^\nu + \frac{i\hbar}{2} \theta^{\mu\nu} \, , \qquad [x^\mu \stackrel{\star}{,} x^\nu] = i\hbar\theta^{\mu\nu}$$

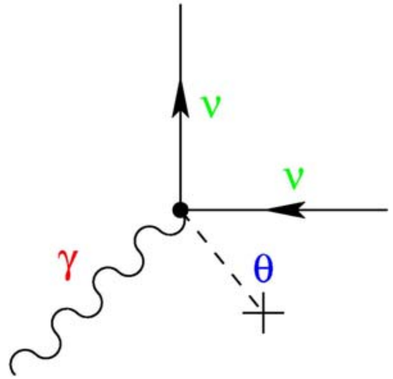
Perturbative effects

Lorentz violation, new interactions, SM forbidden decays

- Plasmon decay: $\gamma_{\text{pl}} \to \nu\bar{\nu}$
  Neutrino-photon interaction from star commutator

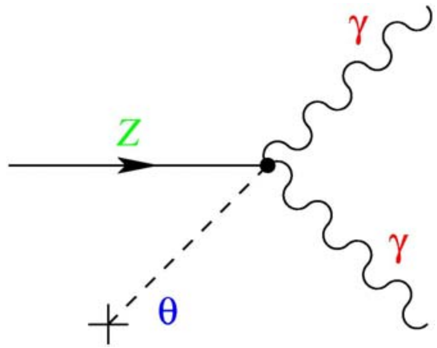
  $$\widehat{D}_\mu \widehat{\psi} = \partial_\mu \widehat{\psi} - ie\widehat{A}_\mu \star \widehat{\psi} + ie\widehat{\psi} \star \widehat{A}_\mu$$

  Neutrino dipole moments



- Gauge sector: $Z \to \gamma\gamma$
  New triple gauge boson interaction

  violates Yang theorem: spin & parity



Non-perturbative aspects

- Ultraviolet-infrared mixing (UV/IR)

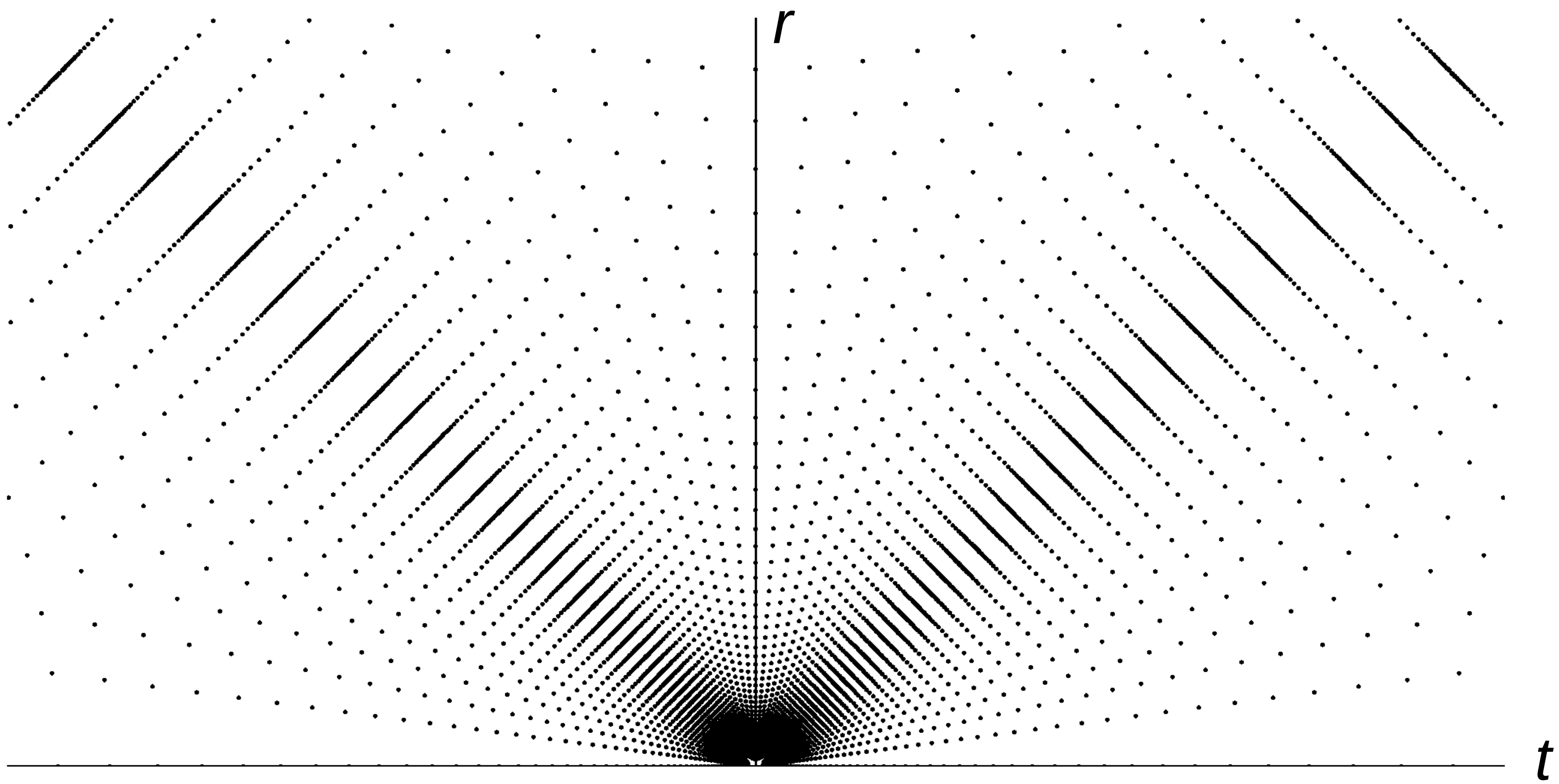
$$\Lambda^2 = |p^\mu \theta^2_{\mu\nu} p^\nu|^{-1} \to \infty \quad \text{for} \quad p \to 0$$

- Non-abelian character of abelian NC gauge theory

$$\beta(g^2) = \frac{\partial g^2}{\partial \ln \Lambda} = -\frac{22}{3}\frac{g^4 N^2}{8\pi^2}$$

(like SU(N) – but formula holds also for *N = 1*)

- Discrete structure of non-commutative spacetime



Spectrum of q-Minkowski space Cerchiai, Wess (1998)

Noncommutative local symmetry

Global versus local symmetries

We should consider *x*-dependent θ:

$$[x^\mu \stackrel{\star}{,} x^\nu] = i\hbar\theta^{\mu\nu}(x)$$

$$f \star g = fg + \frac{i\hbar}{2}\sum_{i,j}\theta^{ij}\,\partial_i(f)\,\partial_j(g) + \frac{-\hbar^2}{4}\sum_{i,j,k,l}\theta^{ij}\theta^{kl}\,\partial_i\partial_k(f)\,\partial_j\partial_l(g) +$$

$$+\frac{-\hbar^2}{6}\left(\sum_{i,j,k,l}\theta^{ij}\,\partial_j(\theta^{kl})\,(\partial_i\partial_k(f)\,\partial_l(g) - \partial_k(f)\,\partial_i\partial_l(g))\right) + O(\hbar^3)$$

Global versus local symmetries

Global symmetries are as in the commutative theory (if θ is transformed as well).

But partial derivatives in the star product still cause problems with local symmetries.

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Global versus local symmetries

Partial derivatives are replaced by covariant derivatives in gauge theories and in general relativity.

But this will i.g. break associativity of the star product.

$$f \star g = fg + \frac{i\hbar}{2} \sum_{i,j} \theta^{ij} \boxed{\partial_i(f)} \partial_j(g) + \frac{-\hbar^2}{4} \sum_{i,j,k,l} \theta^{ij}\theta^{kl} \boxed{\partial_i\partial_k(f)} \partial_j\partial_l(g) +$$

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Global versus local symmetries

Partial derivatives are replaced by covariant derivatives in gauge theories and in general relativity.

But this will i.g. break associativity of the star product.

The problem can be addressed in two ways:

- Noncommutative gauge symmetry
- Noncommutative twisted symmetry

Noncommutative gauge symmetry

Star products are not compatible with ordinary gauge transformations, but they are compatible with NC gauge transformations:

$$D_\mu \psi = \partial_\mu \psi - i A_\mu \star \psi$$

$$\widehat{\delta} D_\mu = i[\Lambda \stackrel{\star}{,} D_\mu], \qquad \widehat{\delta} F_{\mu\nu} = i[\Lambda \stackrel{\star}{,} F_{\mu\nu}],$$

$$\widehat{\delta} \Psi = i \Lambda \star \Psi, \qquad \widehat{\delta} \bar{\Psi} = -i \bar{\Psi} \star \Lambda$$

Diffeomorphisms

Consider two scalar functions f,g
and a vector field $\xi = \xi^\mu(x)\,\partial_\mu$.

Under an infinitesimal coordinate transformation

$$x^\mu \mapsto x^\mu + \xi^\mu : \quad \delta_\xi f = - \xi f, \quad \delta_\xi g = - \xi g ,$$

the star product is not covariant

$$\delta_\xi (f * g) \neq \delta_\xi f * g + f * \delta_\xi g .$$

In fact

$$\xi (f * g) \neq \xi f * g + f * \xi g + f *_{\xi(\theta)} g$$

The simple star product of two
scalar functions *f, g* fails to be covariant:

$$f \star g$$

is not a scalar function.

Instead,

$$\rho(f) \star \rho(g) = \rho(f \star' g)$$

We need to consider equivalence classes of
star products: Gauge theory of the star product?

Noncommutative twisted symmetry

Twisted gauge theory

Covariant derivative (as in NC gauge theory)

$$D_\mu\psi = \partial_\mu\psi - iA_\mu \star \psi$$

Twisted coproduct

$$\Delta_\star\lambda \equiv \sum \lambda_{(1)} \otimes \lambda_{(2)} = \mathcal{F}(\lambda \otimes 1 + 1 \otimes \lambda)\mathcal{F}^{-1}$$

for the transformation of products of fields

$$\delta_\lambda(\phi \star \psi) = \sum (\delta_{\lambda_{(1)}}\phi) \star (\delta_{\lambda_{(2)}}\psi)$$

Twisted gauge theory

Twisted gauge transformation

...of matter fields

$$\delta_\lambda \psi = i\lambda \cdot \psi \equiv i\lambda_a(x) T_a \cdot \psi$$

...of gauge fields

$$\delta_\lambda A_\mu = \partial_\mu \lambda + i\lambda_a [T_a \,; \, A_\mu]$$

...of covariant derivatives

$$\delta_\lambda (D_\mu \psi) = \delta_\lambda (\partial_\mu \psi - iA_\mu \star \psi) = i\lambda \cdot (D_\mu \psi)$$

Tensor fields on NC spacetime

For two tensor fields *V*, *W*

$$V \star W$$

should be a tensor too.

For two scalar functions *f*, *g*

$$f \star g$$

should be a scalar function.

Postulate that $f \star g$ is a scalar:

$$\widehat{\delta}_\xi(f \star g) = -\xi(f \star g)$$

and diffeomorphisms are generated by ordinary vector fields

The vector field ξ does not „see“ the star product;
We need a product rule: the coproduct

$$\Delta_\star \xi \equiv \sum \xi_{(1)} \otimes \xi_{(2)}$$

$$\widehat{\delta}_\xi(f \star g) = \widehat{\delta}_{\xi_{(1)}} f \star \widehat{\delta}_{\xi_{(2)}} g,$$

In the commutative case the coproduct is

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$$

For the Moyal-Weyl star product (θ const.) the coproduct is obtained by an abelian twist:

$$\Delta_\star = \mathcal{F} \Delta \mathcal{F}^{-1}, \qquad \mathcal{F} = \exp(-\frac{i\theta^{ij}}{2} \partial_i \otimes \partial_j)$$

On scalar functions, the twist generates the star product itself:

$$f \star g = \mu \, [\mathcal{F}^{-1}(f \otimes g)]$$

Twisted local symmetry

- Symmetry transformations are undeformed.
- The product rule is deformed by a twist.
- The failure of co-commutativity induces the noncommutativity of fields.

noncommutative/
twisted gravity

noncommutative/ twisted gravity

with Julius Wess

Paolo Aschieri, Christian Blohmann

Marija Dimitrijevic, Frank Meyer

Symmetry with twisted coproduct

Two simple rules:

- The transformation of individual tensors is undeformed

- Tensors must be star-multiplied

Twisted nc differential geometry

Transformation laws

scalar field

$$\widehat{\delta}_\xi \phi = -\xi\phi, \qquad \xi \equiv \xi^\mu(x)\partial_\mu$$

covariant vector

$$\widehat{\delta}_\xi V_\mu = -\xi^\rho \partial_\rho V_\mu - (\partial_\mu \xi^\rho) V_\rho$$

contravariant vector

$$\widehat{\delta}_\xi V^\mu = -\xi^\rho \partial_\rho V^\mu + (\partial_\rho \xi^\mu) V^\rho$$

Twisted nc differential geometry

Tensors

star products of tensors are again tensors

Note that the star product may act non-trivially on tensor indices: To ensure covariance and associativity the derivatives in the twist must be Lie-derivatives.

For the simple Moyal-Weyl star product this does not happen. The following formulas are given for that case.

Twisted nc differential geometry

Connection

covariant derivative

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\alpha_{\mu\nu} \star V_\alpha$$

curvature and torsion

$$[D_\mu, D_\nu] V_\rho = R_{\mu\nu\rho}{}^\sigma \star V_\sigma + T_{\mu\nu}{}^\alpha \star D_\alpha V_\rho$$

$$R_{\mu\nu\rho}{}^\sigma = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\beta_{\nu\rho} \star \Gamma^\sigma_{\mu\beta} - \Gamma^\beta_{\mu\rho} \star \Gamma^\sigma_{\nu\beta}$$

$$T_{\mu\nu}{}^\alpha = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}$$

Noncommutative gravity

metric

an invertible real symmetric tensor $G_{\mu\nu}$
whose covariant derivative vanishes.

Christoffel symbols

$$\Gamma^{\sigma}_{\alpha\beta} = \frac{1}{2}(\partial_\alpha G_{\beta\gamma} + \partial_\beta G_{\alpha\gamma} - \partial_\gamma G_{\alpha\beta}) \star G^{\gamma\sigma\star}$$

Noncommutative gravity

Ricci tensor

$$R_{\mu\nu} = R_{\mu\nu\rho}{}^{\rho}$$

curvature scalar

$$R = G^{\mu\nu\star} \star R_{\mu\nu}$$

action / equations of motion...

$$S = \int d^n x \sqrt{-g} R \ \ (+cc) \qquad\qquad R_{\mu\nu} - \frac{1}{2} R \star G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Noncommutative gravity equations

$$R_{\mu\nu} - \frac{1}{2} R \star G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Solutions?

Formidable task, considering that the noncommuative Einstein equations contain derivatives to all orders.

- Perturbative corrections to known solutions
- Use symmetries

Noncommutative gravity equations

$$R_{\mu\nu} - \frac{1}{2} R \star G_{\mu\nu} = 0$$

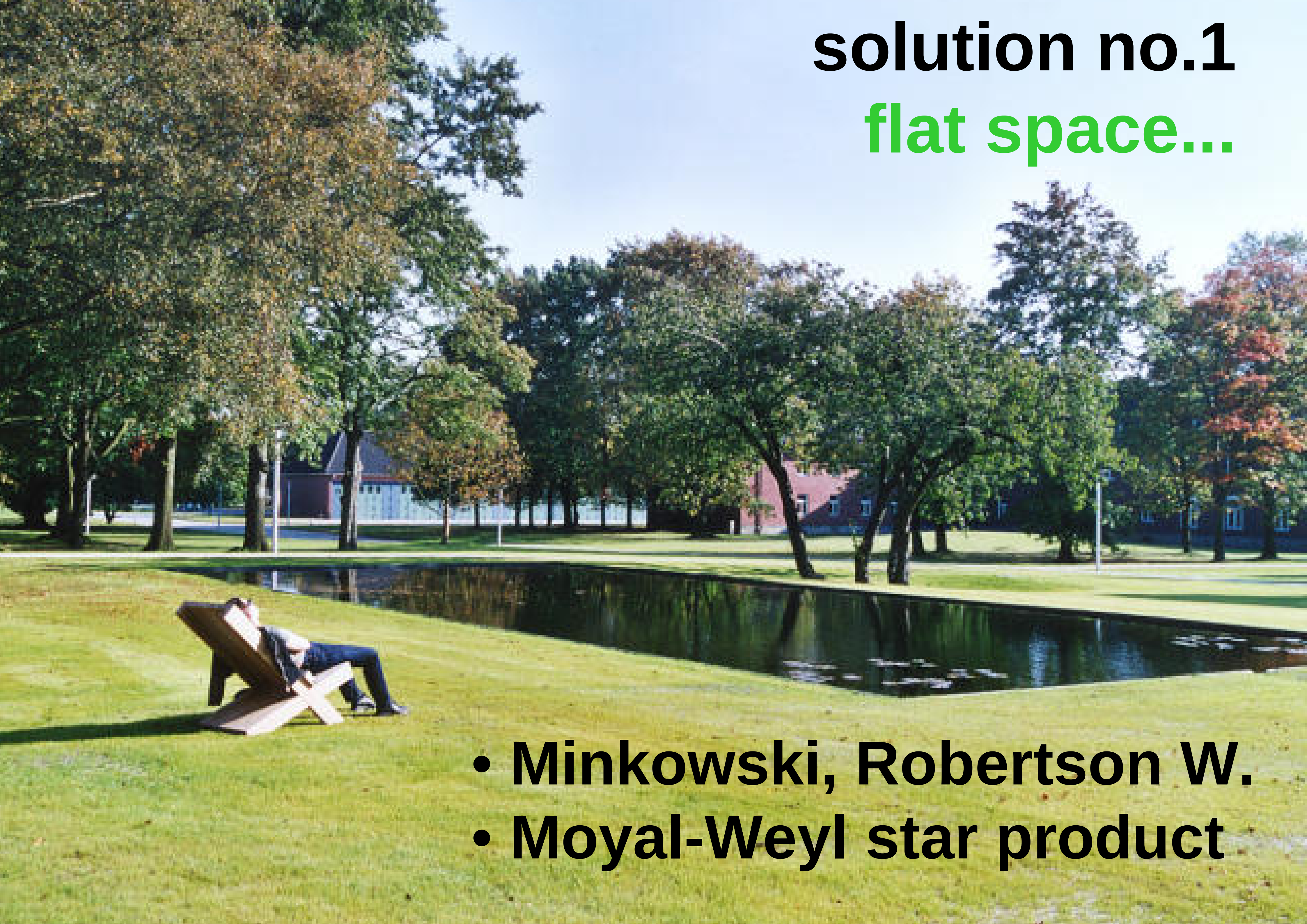
A solution of the NC gravity equation is a mutually compatible pair:

- an algebra / twist $\qquad \star/\mathcal{F}$
- a metric $\qquad g^{\mu\nu}$

noncommutative gravity

exact solotions

with Sergey Solodukhin



solution no.1
flat space...

- Minkowski, Robertson W.
- Moyal-Weyl star product

solution no.2

curved space...

solution no.2

curved space...

...with spherical symmetry

Spherical symmetry

Killing vectors

$$[\xi_i, \xi_j] = i\epsilon_{ijk}\xi_k \qquad \mathcal{L}_{\xi_i} g^{\mu\nu} = 0$$

compatible algebra

$$[x_i \overset{\star}{,} x_j] = 2i\lambda\epsilon_{ijk}x_k$$

note:

- λ may also depend on casimir
- the product acts non-trivially on tensors

Star product

$$f \star g = fg + \sum_{n=1}^{\infty} C_n(\frac{\lambda}{\rho}) \xi_+{}^n f \, \xi_-{}^n g$$

$$\begin{aligned} C_n(\frac{\lambda}{\rho}) &= B(n, \frac{\rho}{\lambda}) \\ &= \frac{\lambda^n}{n! \, \rho(\rho-\lambda)(\rho-2\lambda)\cdots(\rho-(n-1)\lambda)} \end{aligned}$$

Grosse, Presnajder; Alekseev, Lachowska; Sämann...

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projective twist: (Kürkcüoglu, Sämann: „Pseudotwist“)

$$\bar{\mathcal{F}} = \sum C_n(\rho) {\mathcal{L}_{\xi_+}}^n \otimes {\mathcal{L}_{\xi_-}}^n$$

isotropic metric

$$ds^2 = -A(\rho)dt^2 + B(\rho)(dx^2 + dy^2 + dz^2) + C(\rho)d\rho^2$$

$$\rho^2 = g_{ij}x^i x^j = x^2 + y^2 + z^2$$

metric and algebra are compatible:

$$T^{\alpha\ldots\omega} \star g^{\mu\nu} = T^{\alpha\ldots\omega} g^{\mu\nu} = g^{\mu\nu} \star T^{\alpha\ldots\omega}$$

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The components of the Killing vectors are linear in the coordinates

$$\xi_i = \epsilon_{ijk} x_j \partial_k$$

it can be shown that

$$\mathcal{L}_{\xi_i} \partial_\sigma T_{\mu_1 \ldots \mu_k} = \partial_\sigma \mathcal{L}_{\xi_i} T_{\mu_1 \ldots \mu_k} \qquad \mathcal{L}_{\xi_i} \partial_{\sigma_1} \ldots \partial_{\sigma_k} g_{\mu\nu} = 0$$

and hence

$$T^{\alpha \ldots \omega} \star \partial_{\sigma_1} \ldots \partial_{\sigma_k} g^{\mu\nu} = T^{\alpha \ldots \omega} \partial_{\sigma_1} \ldots \partial_{\sigma_k} g^{\mu\nu} = \partial_{\sigma_1} \ldots \partial_{\sigma_k} g^{\mu\nu} \star T^{\alpha \ldots \omega}$$

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The star product completely drops out of the noncommutative Einstein equations.

The coefficients A, B, C can be determined in the usual way and a change of coordinates

$$r = (\rho + a/4)^2/\rho, \quad a = 2M$$

gives the Schwarzschild metric

$$ds^2 = -g(r)dt^2 + dr^2/g(r) + r^2 d\omega^2, \quad g(r) = 1 - a/r$$

but with quantized coordinates

Coordinates should be real – we consider only unitary representations of

$$[x_i \overset{\star}{,} x_j] = 2i\lambda\epsilon_{ijk}x_k$$

These are the usual angular momentum irreducible representations:

$$\vec{x}^{\star 2}|j,m\rangle = (2\lambda)^2 j(j+1)|j,m\rangle, \quad 2j = 0,1,2,\ldots$$

In terms of

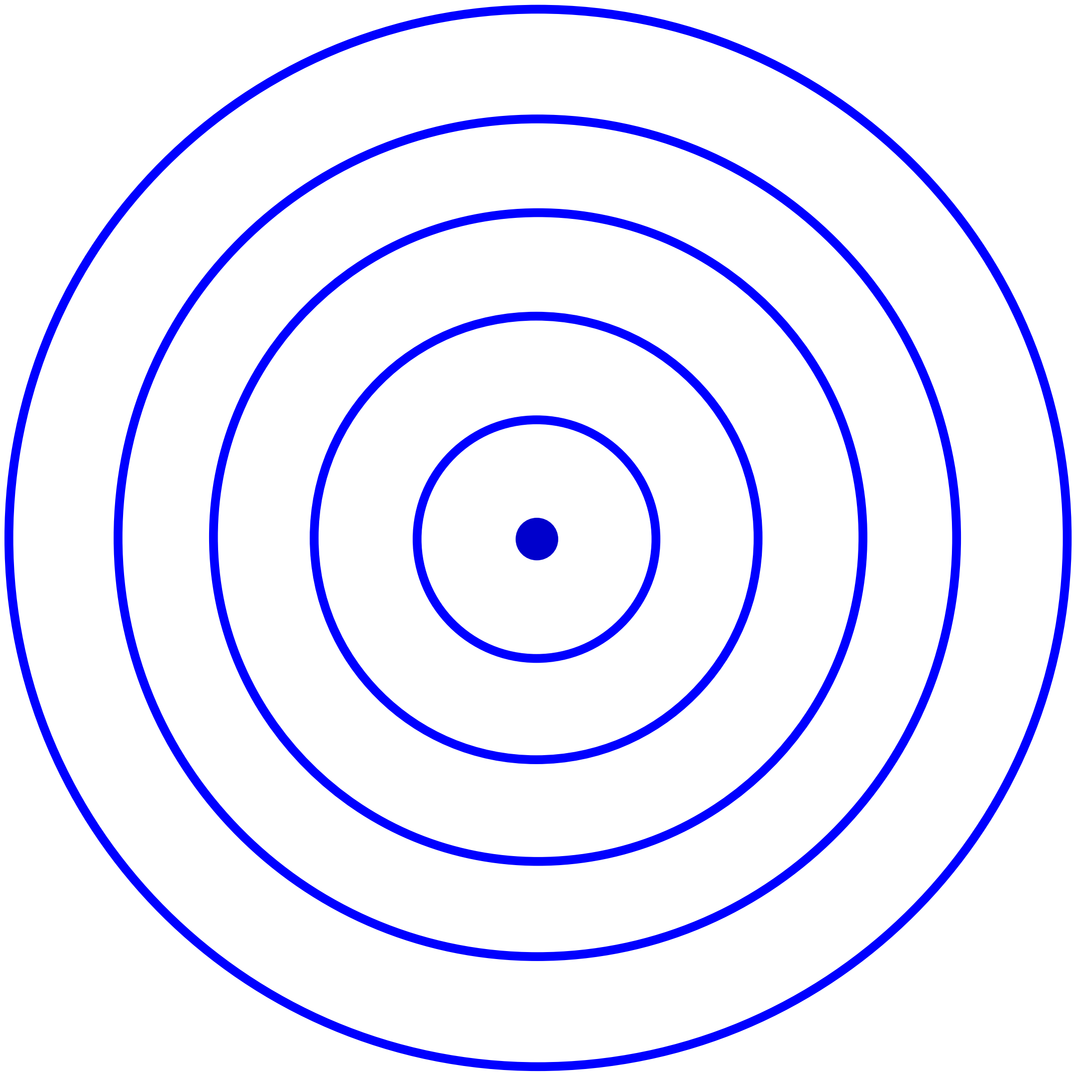
$$\rho^2 = g_{ij} x^i x^j = x^2 + y^2 + z^2$$

we find

$$(\vec{x})^{\star 2} \equiv \sum x_i \star x_i = \rho(\rho + 2\lambda)$$

irreps: $\boxed{\rho = 2j\lambda = n\lambda; \quad n = 0, 1, 2, \ldots}$

schematically (in isotropic coordinates):



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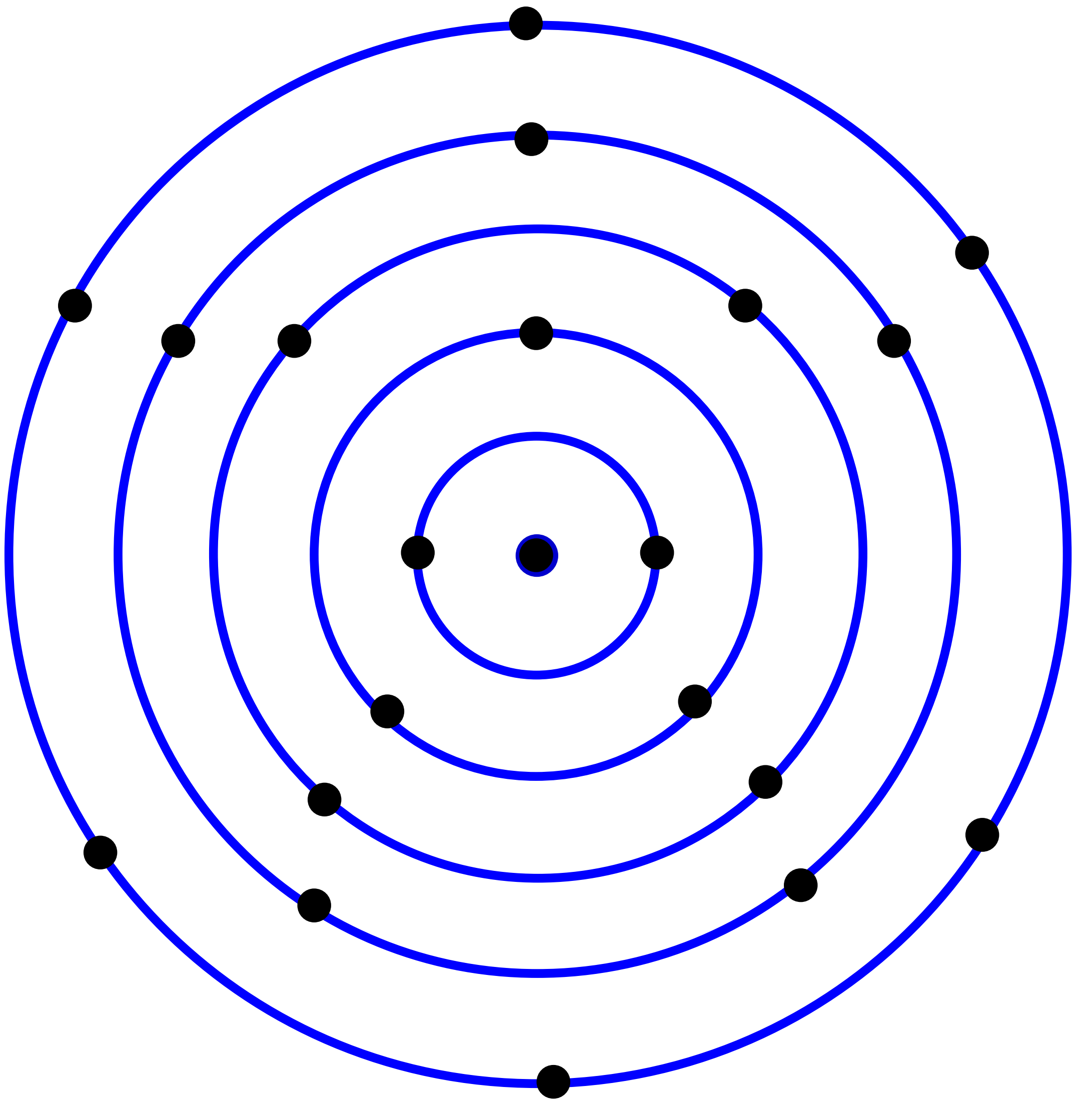

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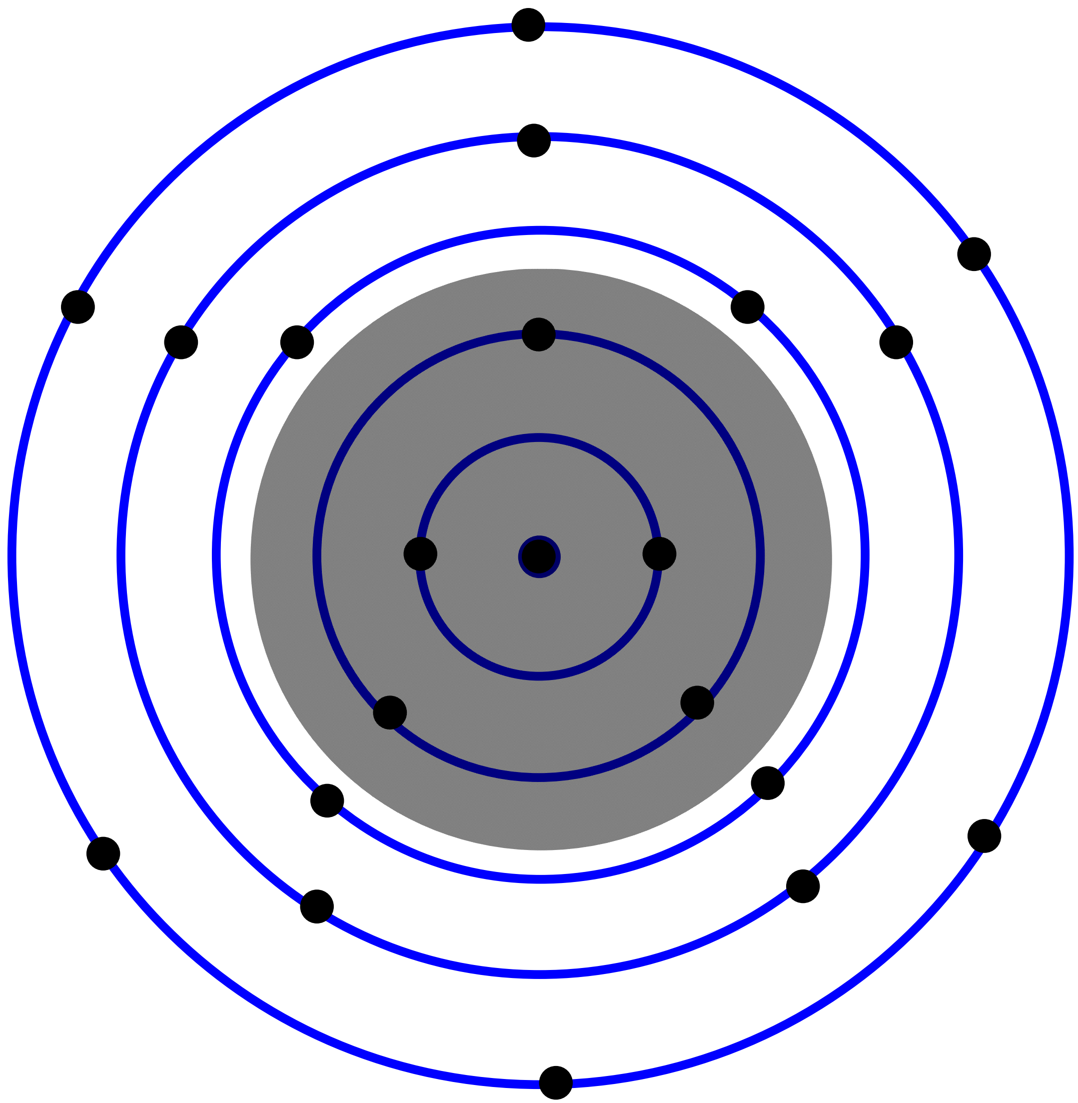

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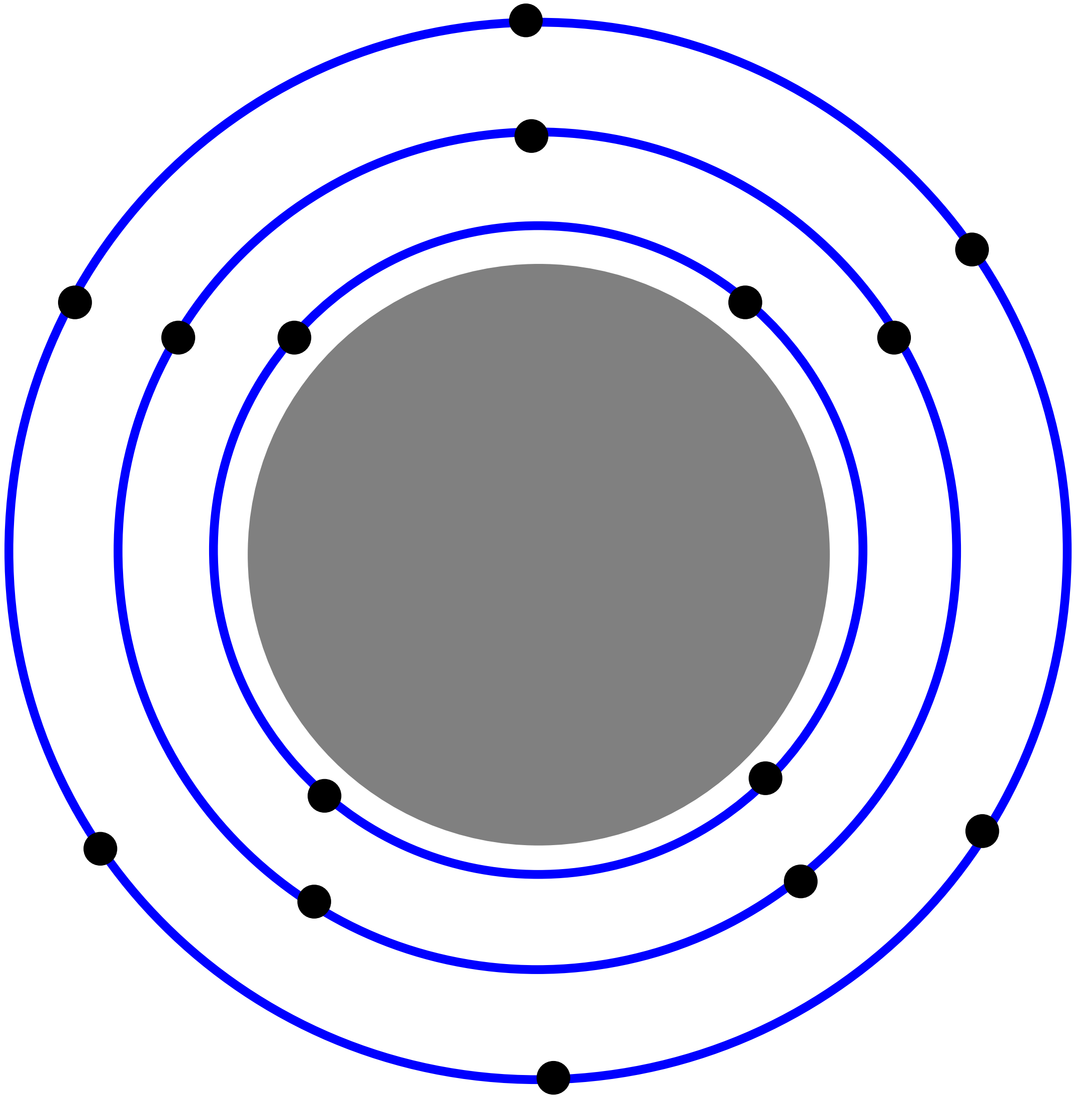
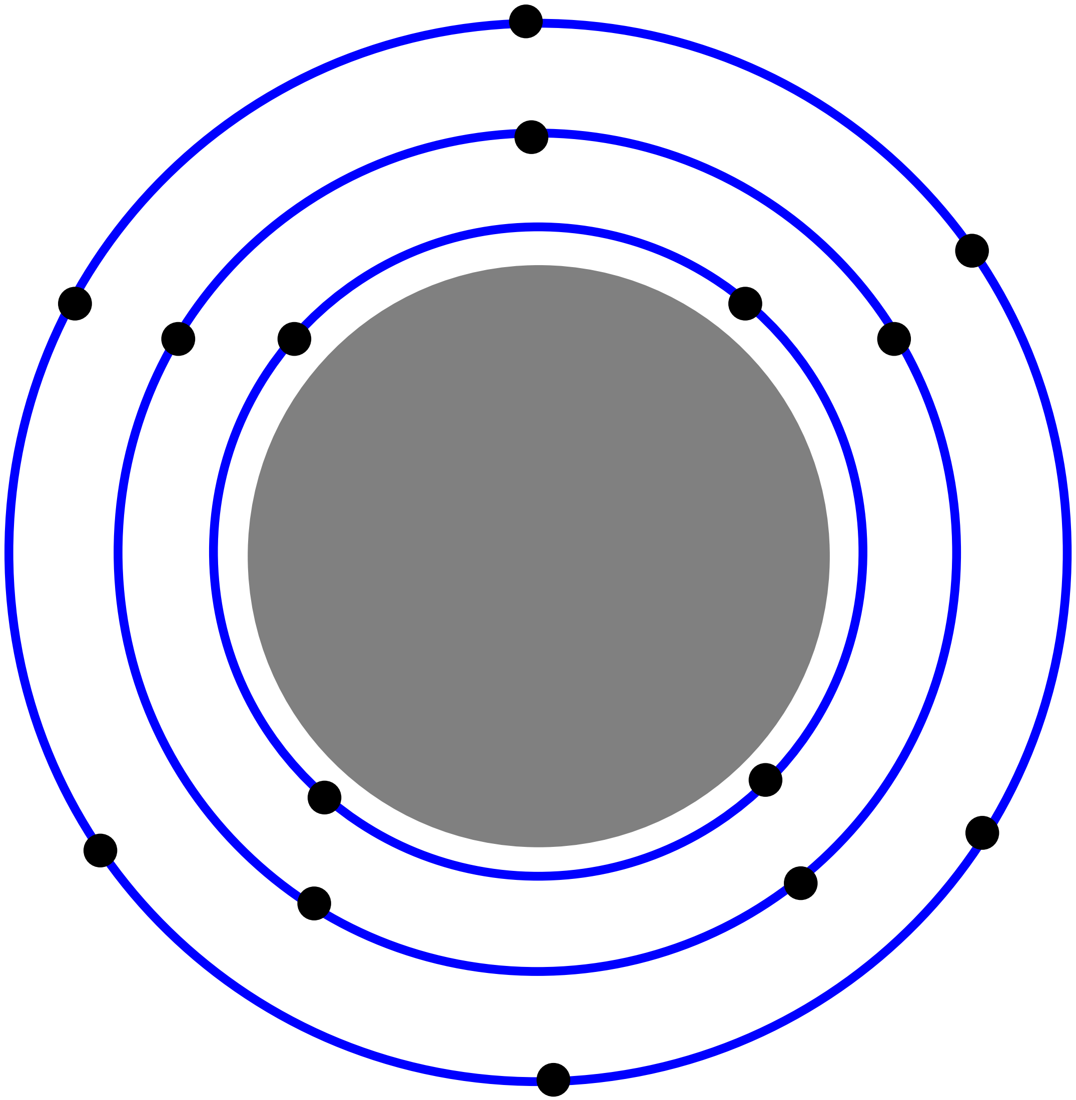
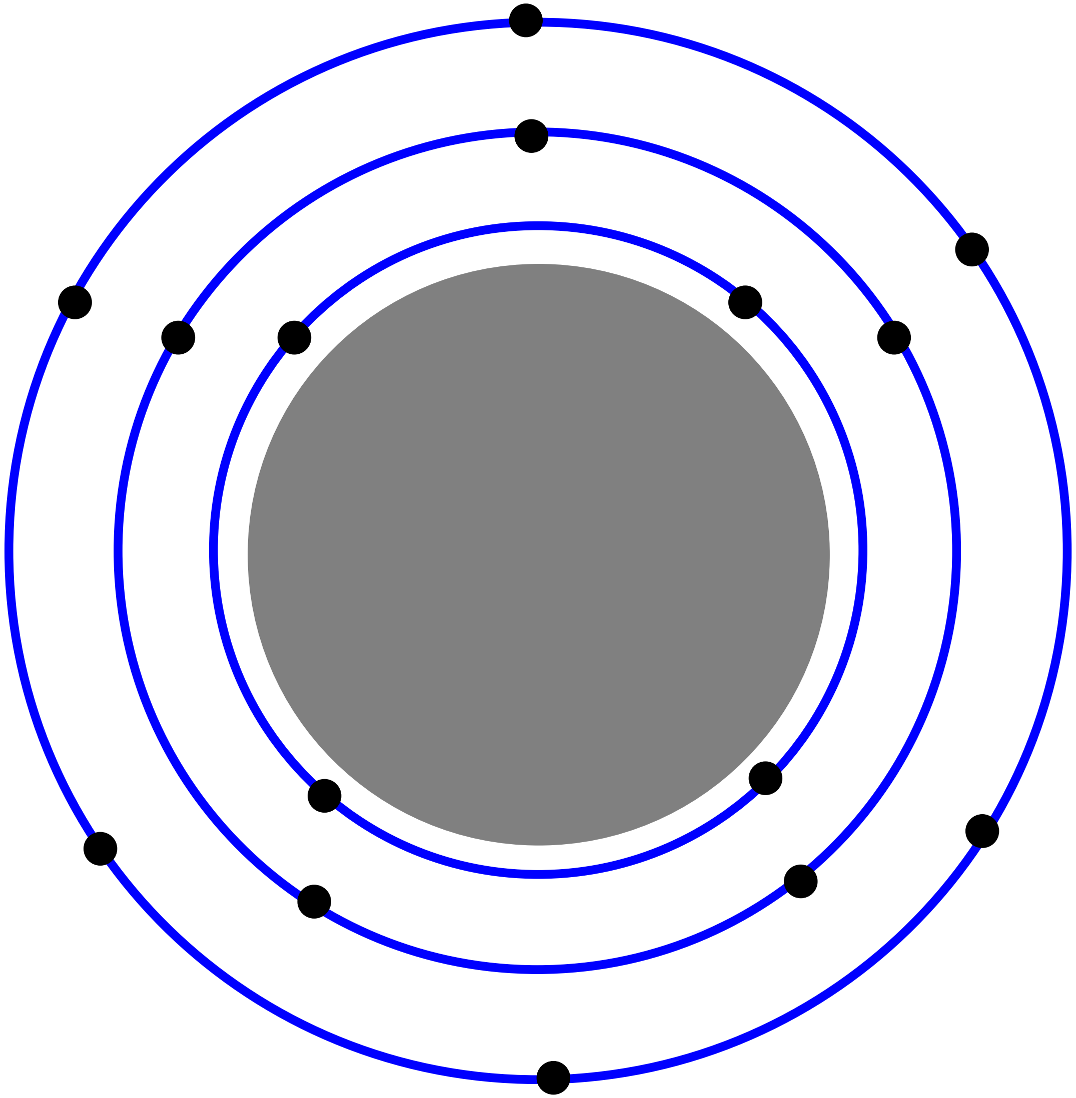


nc black hole = sphere minus fuzzy sphere



„fuzzy black hole“



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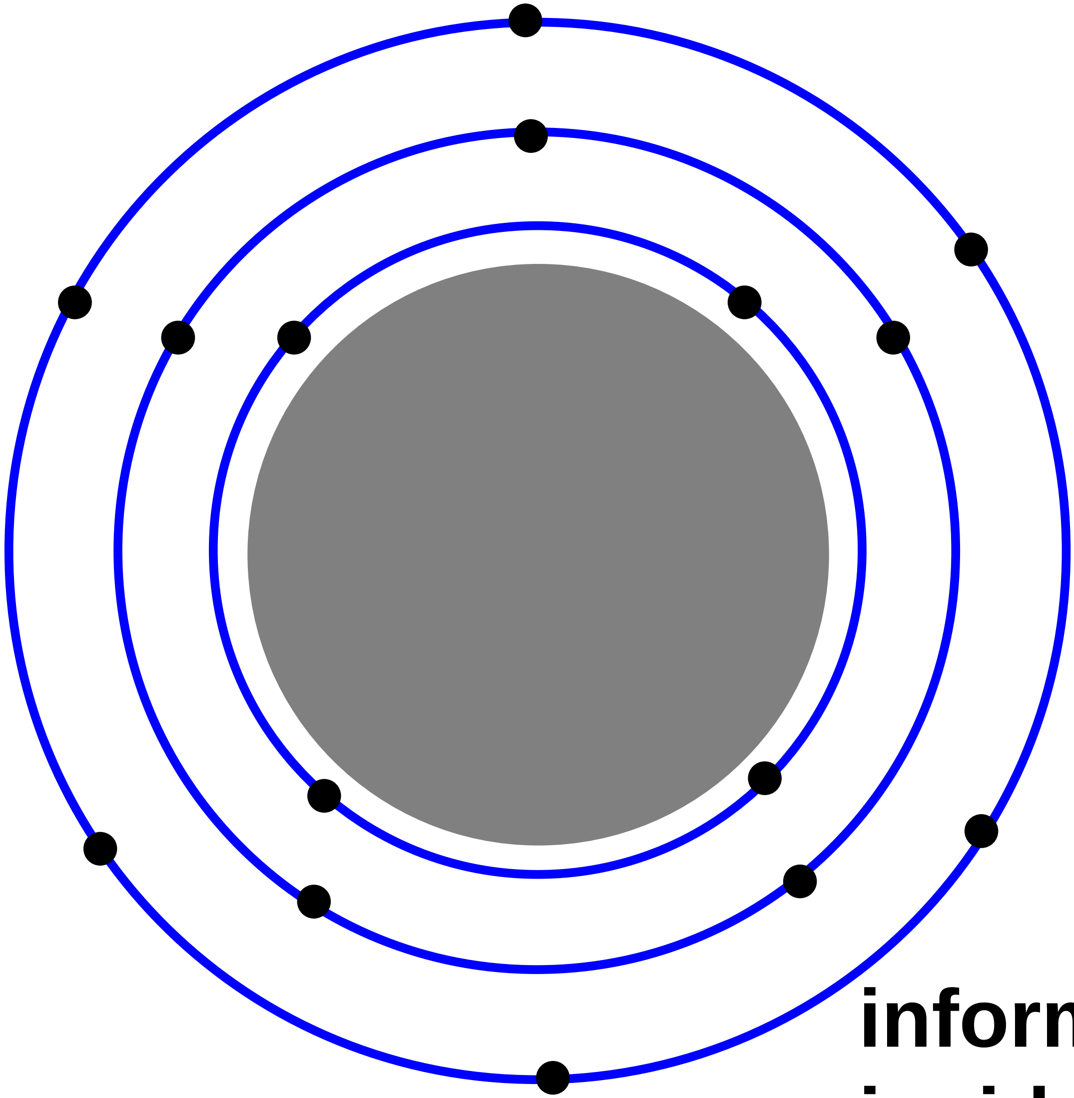


information hidden inside the horizon scales with its area

Inside the horizon...

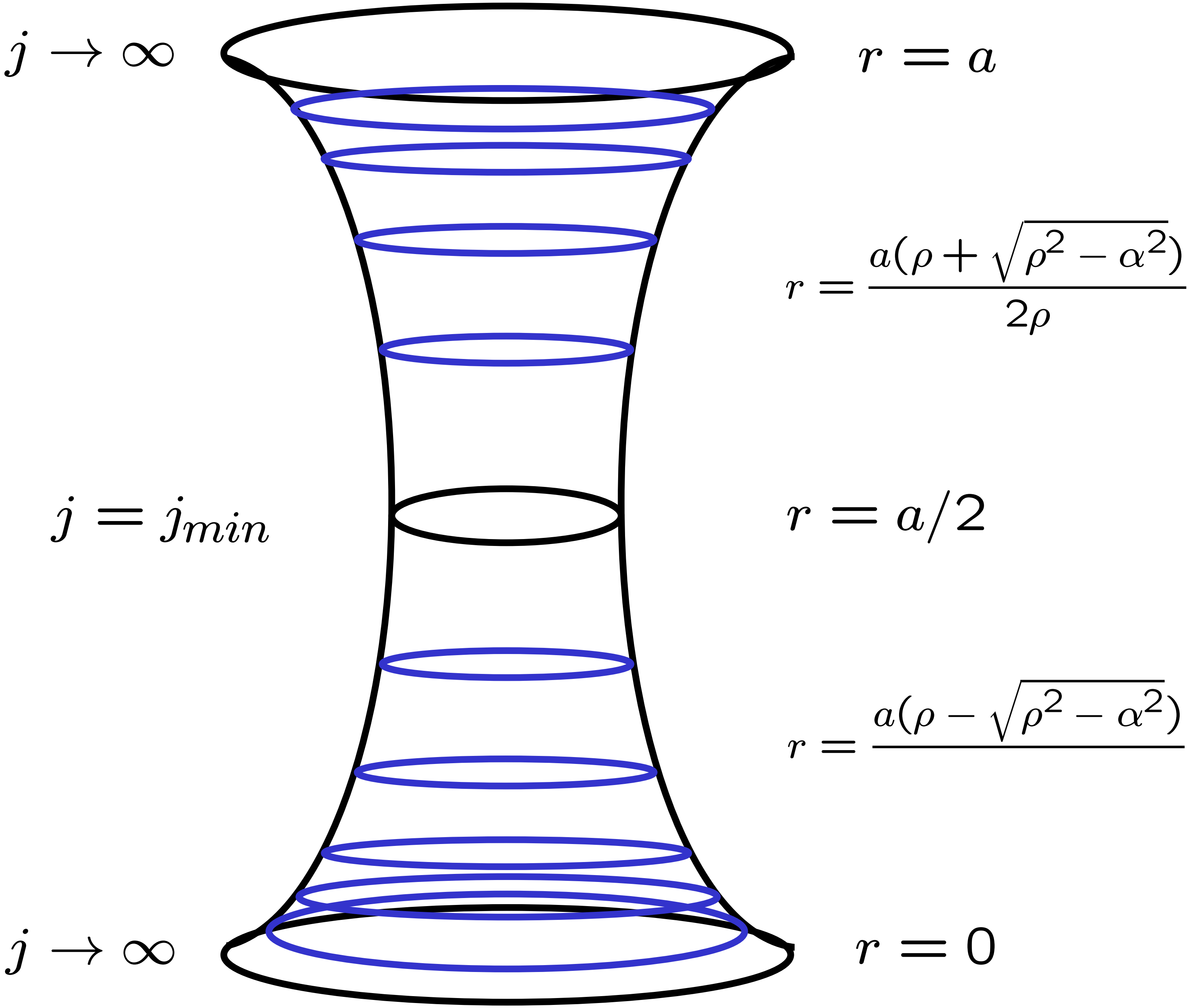
slices of constant time are no longer conformal to Euclidean flat space-time, so the fuzzy sphere construction cannot be used directly

constant time slices are conformal to de Sitter space-time

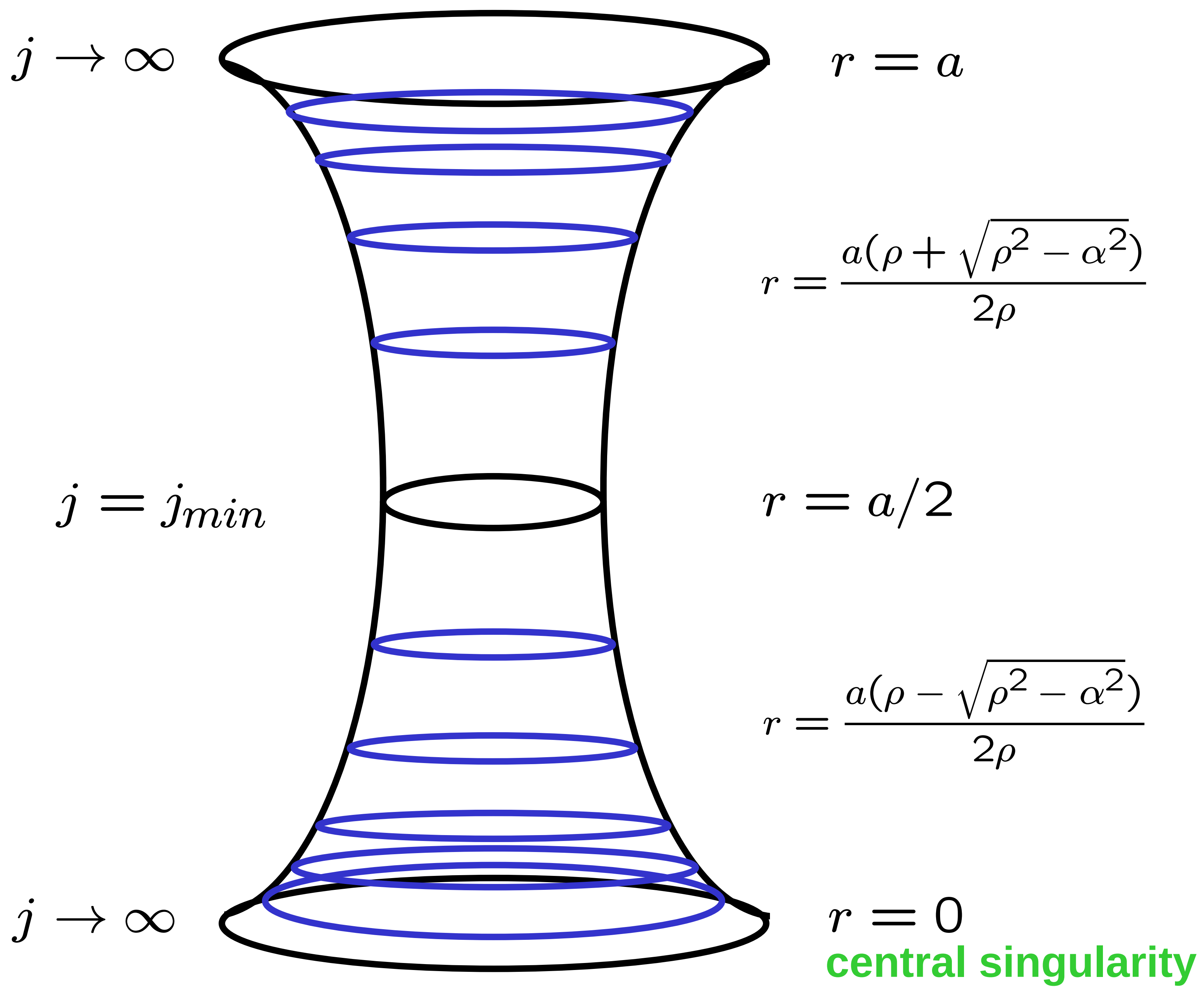
$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$$

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = \alpha^2 > 0$$

we can now quantize $x_1,\ x_2,\ x_3$ as before ...
two fuzzy sphere copies are needed



$j \to \infty$
$r = a$
$r = \frac{a(\rho + \sqrt{\rho^2 - \alpha^2})}{2\rho}$
$j = j_{min}$
$r = a/2$
$r = \frac{a(\rho - \sqrt{\rho^2 - \alpha^2})}{}$
$j \to \infty$
$r = 0$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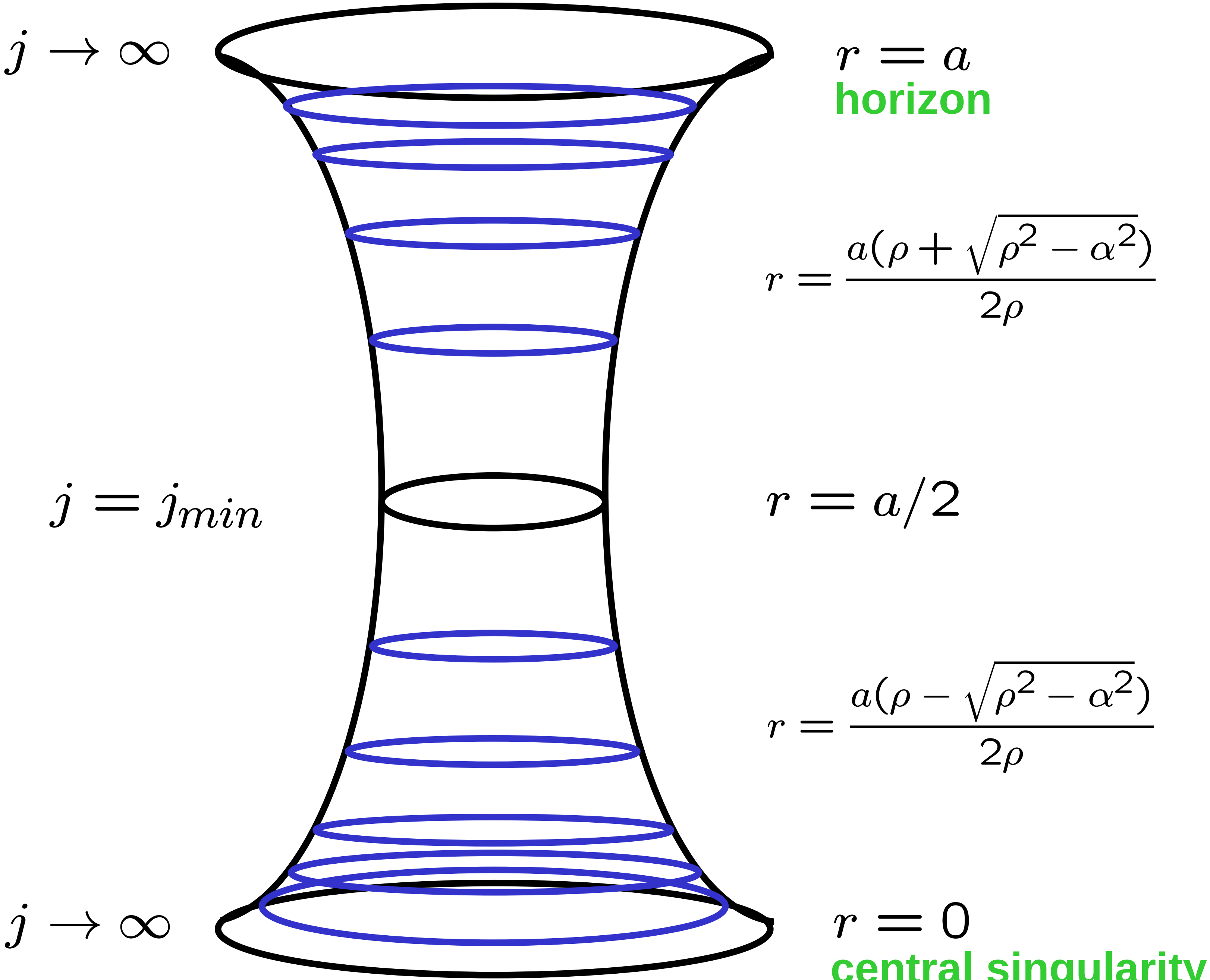
$r = a/2$
$r = \frac{a(\rho - \sqrt{\rho^2 - \alpha^2})}{2\rho}$
$j \to \infty$
$r = 0$
central singularity


$j \to \infty$
$r = a$
horizon
$r = \frac{a(\rho + \sqrt{\rho^2 - \alpha^2})}{2\rho}$
$j = j_{min}$
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$r = \frac{a(\rho - \sqrt{\rho^2 - \alpha^2})}{2\rho}$
$j \to \infty$
$r = 0$
central singularity

Summary and Outlook

noncommutative gravity

- simple covariant construction via twisted tensor calculus
- dynamics of noncommutativity?

fuzzy black hole

- spherically symmetric solution
- discrete, quasi-2D onion-type spacetime
- entropy naturally scales with area
- quantization of mass?

Thank you